UNIT V – Z- TRANSFORMS AND DIFFERENCE EQUATIONS

PART –A

1. **Define Z-Transform**

ANS

$$Z[f(n)]=\sum_{n=0}^{\infty} f(n).z^{-n} = F[z]$$

2. **Find the Z-Transform of a^n.**

ANS

$$\text{WKT } Z[f(n)]=\sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[a^n]=\sum_{n=0}^{\infty} a^n z^{-n}=\sum_{n=0}^{\infty}\frac{a^n}{z^n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n=1+\left(\frac{a}{z}\right)+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\cdots=\left(1-\frac{a}{z}\right)^{-1}=\left(\frac{z-a}{z}\right)^{-1}=\frac{z}{z-a}$$

3. **Find Z[n]**

ANS

$$\text{WKT } Z[f(n)]=\sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[n]=\sum_{n=0}^{\infty} n.z^{-n}=\sum_{n=0}^{\infty}\frac{n}{z^n}$$

$$=\frac{0}{z^0}+\frac{1}{z^1}+\frac{2}{z^2}+\frac{3}{z^3}+\cdots=\frac{1}{z^1}+\frac{2}{z^2}+\frac{3}{z^3}+\cdots$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right)=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}=\frac{1}{z}\left(\frac{z}{z-1}\right)^2=\frac{z}{(z-1)^2}$$

4. **Find the Z-Transform of** $\sin\left(\frac{n\pi}{2}\right)$

ANS

$$\text{w.k.t } Z[a^n.\sin(n\theta)]=\frac{za\sin\theta}{z^2-2za\cos\theta+a^2}$$

$$\text{Put } a=1 \text{ and } \theta=\frac{\pi}{2}$$

$$\therefore Z\left[\sin\left(\frac{n\pi}{2}\right)\right]=\frac{z}{z^2+1}$$

5. **Define unit step sequence. Write its z-Transform**

ANS

$$u(n)=\begin{cases}1, n\ge 0\\ 0, \text{otherwise}\end{cases}$$

$$Z[u(n)]=\sum_{n=0}^{\infty}u(n).z^{-n}=\sum_{n=0}^{\infty}1.z^{-n}=1+z^{-1}+z^{-2}+z^{-3}+...=1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...=\frac{z}{z-1}$$

$$\therefore Z[u(n)]=\frac{z}{z-1}$$

6. **State Convolution theorem on Z-Transform**

ANS

$Z[f(n)*g(n)]=Z[f(n)].Z[g(n)]$ where $f(n)*g(n)=\sum_{k=0}^{n}f(k).g(n-k)$

7. **State and prove Initial value theorem on Z-Transform**

ANS

If $Z[f(n)]=F[z]$, then $f(0)=\lim_{z\to\infty}F[z]$

Proof

$$\text{w.k.t } Z[f(n)]=\sum_{n=0}^{\infty}f(n).z^{-n}=f(0)+f(1).z^{-1}+f(2).z^{-2}+f(3).z^{-3}+....$$

$$F[z]=f(0)+\frac{f(1)}{z}+\frac{f(2)}{z^2}+\frac{f(3)}{z^3}+...$$

$$\lim_{z\to\infty}F[z]=f(0)$$

8. **Obtain** $Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

ANS

$$\text{Let}F[z]=\frac{z}{(z+1)(z+2)}$$

$$z^{n-1}F[z]=\frac{z^n}{(z+1)(z+2)}$$

$z=-1$ is a simple pole. $z=-2$ is also a simple pole.

$$\underset{z=-1}{\text{Res}}=\lim_{z\to-1}\frac{z^n}{(z+2)}=(-1)^n.$$

$$\underset{z=-2}{\text{Res}}=\lim_{z\to-2}\frac{z^n}{(z+1)}=-(-2)^n.$$

$$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]=(-1)^n-(-2)^n.$$

9. **Form a difference equation by eliminating arbitrary constants from $u_n=A.2^{n+1}$**

ANS

$u_n = A.2^{n+1}$

$u_{n+1} = A.2^{n+1+1} = A.2^{n+1}.2 = 2u_n$

The difference equation is $u_{n+1} = 2u_n$.

10. **Find the difference equation generated by** $y_n = a.n + b.2^n$

ANS

$y_n = a.n + b.2^n$

$y_{n+1} = a.(n+1) + b.2^{n+1}$

$y_n = a.(n+2) + b.2^{n+2}$

$$\text{The difference equation is} \begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix} = 0.$$

On expanding $2ny_n + (2-3n)y_{n+1} + (n-1)y_{n+2} = 0$

11. **What advantage is gained when Z-Transform is used to solve difference equation?**

ANS

Z-Transform converts difference equation to algebraic equation.

12. **Solve y_{n+1}-2y_n=0 given y_0=3.**

ANS

Given $y_{n+1}-2y_n=0$

Apply Z-Transform on both sides, $Z[y_{n+1}]-2Z[y_n]=Z[0]$

$z.Z[y_n]-z.y_0-2.Z[y_n]=0$

$(z-2).Z[y_n]-3z=0$

$$Z[y_n] = \frac{3z}{(z-2)}$$

$$y_n = Z^{-1}\left[\frac{3z}{(z-2)}\right] = 3Z^{-1}\left[\frac{z}{(z-2)}\right] = 3.2^n$$

PART – B

1. Find the Z-transform of $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$.
2. Find $Z^{-1}\left[\frac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$
3. Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$
4. Find $Z^{-1}\left[\frac{z^3+3z}{(z-1)^2(z^2+1)}\right]$
5. Using convolution theorem, find the following

1. $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ 2. $Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]$ 3. $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ 4. $Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]$ 5. $Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right]$

6. $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

6. Find the inverse Z- Transform of $\frac{10z}{z^2-3z+2}$ using Residue method.
7. Find the Inverse Z-Transform of $\frac{z(z+1)}{(z-1)^3}$ using Residue method.
8. Solve $y_{n+2}+y_n=2$ given y_0=0 and y_1=0 by using Z-Transforms.
9. Solve $y_{n+2}+6y_{n+1}+9y_n=2^n$ given that y_0=0 and y_1=0.
10. Solve $u_{n+2}+4u_{n+1}+3u_n=3^n$ given that u_0=0 and u_1=1.