Unit - 5

Boundary Value Problems for ordinary & PDE

(X) (1) Obtain the finite difference scheme for the differential equation $2y'' + y = 5$

Soln

$$y'' + \frac{y}{2} = \frac{5}{2} \quad \text{---} \quad (1)$$

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$(1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{y_i}{2} = \frac{5}{2}$$

$$y_{i-1} - 2y_i + y_{i+1} + \frac{1}{2}h^2 y_i = \frac{5}{2}h^2$$

$$y_{i-1} - \left[2 - \frac{h^2}{2}\right] y_i + y_{i+1} = \frac{5}{2}h^2$$

$$2y_{i-1} - [4 - h^2]\, y_i + 2y_{i+1} = 5h^2$$

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(2) Write Liebmann's iteration process.

$$u_{i,j}^{n+1} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n+1}\right]$$

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(3) Write the diagonal five point formula for solving the two dimensional Laplace equation $\nabla^2 u = 0$.

$$u_{ij} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j-1} + u_{i-1,j+1}\right]$$

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(4) Using finite difference solve $y'' - y = 0$ given $y(0) = 0$, $y(1) = 1$, $n = 2$.

$$y'' - y = 0 \quad \text{---} \quad (1)$$

$$h = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}$$

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

① $\Rightarrow$ $\dfrac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - y_i = 0$

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{1/4} - y_i = 0$$

$$4(y_{i+1} + y_{i-1} - 2y_i) - y_i = 0$$

$$4y_{i+1} + 4y_{i-1} - 9y_i = 0$$

$$y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0$$

$i = 1$, $y_2 - \frac{9}{4} y_1 + y_0 = 0$ $\quad [\because i = 1,\ y_2 = 1]$

$$1 - \frac{9}{4} y_1 + 0 = 0$$

$$\frac{9}{4} y_1 = 1$$

$$y_1 = \frac{4}{9}$$

$$y_1 = 0.4444$$

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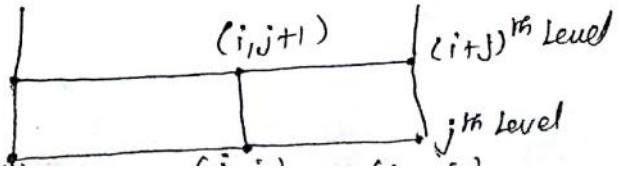
⑤ State whether the Crank Nicholson's Scheme is an explicit or implicit Scheme Justify.

The crank Nicholson's Scheme is implicit Scheme. The Schematic representation of Crank Nicholson's Method is shown below.



The Solution Value at any point $(i, j+1)$ on the $(j+1)^{th}$ level is dependent on the Solution Values at the neighbouring points on the same level and on three values on the j^{th} level. Hence it is an implicit Method.

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(6) Write the finite difference approximations of $y'(x)$ & $y''(x)$.

$$y'(x) = \frac{1}{2h}[y_{i+1} - y_{i-1}]$$

$$y''(x) = \frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

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(7) State Standard five point formula.

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}]$$

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(8) What is the central difference approximation for y''?

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

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(9) State crank Nicholsons difference Scheme?

$$u_t = \alpha^2 u_{xx}$$

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

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(10) Write down Bender-Schmidt's difference scheme in general form and using suitable value of λ, write the scheme in simplified form.

$$u_{xx} - a u_t = 0$$

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i-1,j}$$

where $\lambda = \frac{k}{h^2 a}$, h is the space for the variable x and k is the space in the time direction.

Unit - V

Boundary Value Problems In Ordinary And Partial Differential Equations

Part-B

D) Solve $(1+x^2)y'' + 4xy' + 2y = 2$ given that $y(0) = 0, y(1) = \frac{1}{2}$ (take $h = \frac{1}{3}$)

Hints:

$x_0 = 0$	$x_1 = 1/3$	$x_2 = 2/3$	$x_3 = 1$
$y_0 = 0$	$y_1 = ?$	$y_2 = ?$	$y_3 = 1/2$

Rewrite the Equation as $(1+x_i^2)y_i'' + 4x_i y_i' + 2y_i = 2$.

Substitute $y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ & $y_i' = \frac{y_i - y_{i-1}}{h}$ and

-Simplify we get

$$9(1+x_i^2)(y_{i+1} - 2y_i + y_{i-1}) + 12x_i(y_i - y_{i-1}) + 2y_i = 2.$$

Put $i = 1$ we get $9(1+x_1^2)(y_2 - 2y_1 + y_0) + 12x_1(y_1 - y_0) + 2y_1 = 2$

$\Rightarrow -26y_1 + 16y_2 = 2$

$\div (-26) \Rightarrow y_1 - 0.6154y_2 = -0.0769 \rightarrow (1)$

Put $i = 2$ we get $9(1+x_2^2)(y_3 - 2y_2 + y_1) + 12x_2(y_2 - y_1) + 2y_2 = 2$

$\Rightarrow 17y_1 - 40y_2 = -10.5$

$\div (17) \Rightarrow y_1 - 2.3529y_2 = -0.61765 \rightarrow (2)$

Solving (1) and (2) we get $y_1 = 0.11455$ and $y_2 = 0.3112$.

(ii) Using the finite difference method, solve $y'' + y = x$ subject to $y(0) = 0$, $y(1) = 2$ at 0.25, 0.5 & 0.75

Hints:

Rewrite the equation as $y_i'' + y_i = x_i$ & $h = 0.25 = \frac{1}{4}$

Substitute $y_i'' = \frac{y_{i+1} - 2y + y_{i-1}}{h^2}$ & $y_i' = \frac{y_i - y_{i-1}}{h}$ and simplify we get

$$y_{i-1} - y_i(2 - h^2) + y_{i+1} = h^2 x_i$$

$$y_{i-1} - \frac{31}{16} y_i + y_{i+1} = \frac{1}{16} x_i$$

Put $i = 1$ we get

$$y_0 - 1.9375 y_1 + y_2 = 0.0625 x_1$$

$$0 - 1.9375 y_1 + y_2 = 0.0625 (0.25)$$

$$-1.9375 y_1 + y_2 = 0.0156$$

Put $i = 2$ we get

$$y_1 - 1.9375 y_2 + y_3 = 0.0625 x_2$$

$$y_1 - 1.9375 y_2 + y_3 = 0.0625 (0.5)$$

$$y_1 - 1.9375 y_2 + y_3 = 0.0313 \quad (2)$$

Put $i = 3$ we get

$$y_2 - 1.9375 y_3 + y_4 = 0.0625 x_3$$

$$y_2 - 1.9375 y_3 + 2 = 0.0625 (0.75)$$

$$y_2 - 1.9375 y_3 = -1.9531 \quad (3)$$

Solving (1), (2) and (3) we get $y_1 = 0.5443$, $y_2 = 1.0701$ and $y_3 = 1.5604$.

3) Solve $u_{xx} = u_t$ given that $u(x,0) = x(4-x)$, $u(0,t) = 0$, $u(4,t) = 0$ by Bender schmidt is formula compute u up to 5 times steps. (taking $h = 1, k = 1$)

Hints:

Bender schmidt's formula is $u_{i,j+1} = \lambda(u_{i+1,j} + u_{i-1,j}) + (1-2\lambda)u_{i,j}$

(i.e) $E = \lambda(A+C) + (1-2\lambda)B$

A	B	C
D	E	F

Here $a^2 = 1, h = 1, k = 1$

$\therefore \lambda = \frac{ka^2}{h^2} = 1$

Put $\lambda = 1$ in above formula we get $u_{i,j+1} = (u_{i+1,j} + u_{i-1,j}) - u_{i,j}$

(i.e) $E = (A+C) - B$

t \ x	0	1	2	3	4
0	0	3	4	3	0
1	0	1	2	1	6
2	0	1	0	1	0
3	0	-1	2	-1	0
4	0	3	-4	3	0
5	0	-7	10	-7	0

4) Solve $\frac{1}{4} u_t = u_{xx}$, given that $u(x,0) = \frac{x}{2}(8-x)$, $u(0,t) = 0$, $u(6,t) = 6$ by Bender Schmidt's formula compute u up to 5 steps.

Hints:

Here $a^2 = 4$. Since h and k are not known then we use simplest form of Bender schmidt's formula. For $h = \frac{1}{2}$ we get the simplest form. Take h and k values such that $\lambda = \frac{1}{2}$ (i.e) $\frac{ka^2}{h^2} = \frac{1}{2}$.

A B C

D E F

Take $h = 1$ and $k = \frac{1}{8}$. Simplest form of Bender Schmidt's

$$u_{i,j+1} = \frac{(u_{i+1,j} + u_{i-1,j})}{2} \quad \text{(i.e)} \quad E = \frac{(A+C)}{2}$$

t \ x	0	1	2	3	4	5	6
0	0	3.5	6	7.5	8	7.5	6
1/8	0	3	5.5	7	7.5	7	6
1/4	0	2.75	5	6.5	7	6.75	6
3/8	0	2.500	4.625	6.000	6.625	6.500	6
4/8	0	2.313	4.250	5.625	6.250	6.313	6

5) Solve $u_t = u_{xx}$ in $0 < x < 5$, $t \geq 0$ given that $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$, with $h = 1$ by Crank-Nicholson method.

Crank Nicholson formula is

$$\lambda(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1}) = 2[(\lambda-1)u_{i,j} + (\lambda+1)u_{i,j+1}]$$

$$\lambda(A+C+D+F) = 2[(\lambda-1)B + (\lambda+1)E]$$

A	B	C
D	E	F

Here $a = a^2 = 1$ Choose $\lambda = 1$; $\lambda = \frac{K}{ah^2}$

$h = 1$ gives $K = 1$

Put $\lambda = 1$ in above formula

we get. $u_{i,j+1} = \frac{1}{4}(u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j})$

(i.e.) $(A+C+D+F)/4 = E$

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

To find u_1 : Apply formula we get $4u_1 - u_2 = 20$ ..... (1)

To find u_2 : Apply formula we get $u_1 - 4u_2 + u_3 = -40$ ..... (2)

To find u_3 : Apply formula we get $u_2 - 4u_3 + u_4 = -40$ ..... (3)

To find u_4 : Apply formula we get $u_3 - 4u_4 = -220$ ---- (4)

Solving (1), (2), (3) and (4) we get $u_1 = 10.05$, $u_2 = 20.2$, $u_3 = 30.72$, $u_4 = 62.68$.

6) Solve $u_{xx} = 32 u_t$, $h = 0.25$ for $t \geq 0$, $0 < x < 1$, $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = t$.

Hints: The range for x is $(0, 1)$; $h = 0.25$ and $a = 32$

$$k = \frac{ah^2}{2} = 1$$

A	B	C
D	E	F

$$u_{i,j+1} = \frac{(u_{i+1,j} + u_{i-1,j})}{2} \quad \text{(i.e.) } E = \frac{(A+C)}{2}$$

i \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	0.5	2
3	0	0	0.25	1	3
4	0	0.125	0.5	1.625	4
5	0	0.25	0.875	2.25	5

7) Solve $4u_{tt} = u_{xx}$, given that $u(x,0) = x(4-x)$, $u(0,t) = 0$, $u(4,t) = 0$ and $u_t(x,0) = 0$ by Explicit formula (Take $h = 1$ and upto $t = 4$)

Hints:

Explicit formula is $u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$

The given wave equation is $4u_{tt} = u_{xx}$

$u_{tt} = \frac{1}{4} u_{xx}$

$u_{tt} = a^2 u_{xx}$

$a^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2}$

Let $h = 1$, $k = \frac{h}{a} = \frac{1}{(1/2)} = 2$

$t = 0, 2, 4$

$h = 0, 1, 2, 3, 4$

$u_{i,j+1} = (u_{i+1,j} + u_{i-1,j}) - u_{i,j}$ (i.e) $H = (D+F) - B$

Condition $u_t(x,0) = 0$ is equivalent to $E = \frac{(A+C)}{2}$

t \ x	0	1	2	3	4
0	0	3	4	3	0
2	0	2	3	2	0
4	0	0	0	0	0

8) Solve $25u_{xx} - u_{tt} = 0$, given $u(0,t) = 0 = u(5,t)$, $u_t(x,0) = 0$, $u(x,0) = \begin{cases} 2x & \text{for } 0 \le x \le 2.5 \\ 10-2x & \text{for } 2.5 \le x \le 5 \end{cases}$.

Compute u up to one half Period of oscillation, taking $h=1$.

Hints:

Explicit formula is $u_{i,j+1} = 2(1-\lambda^2 a^2)u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$

(i.e) $H = 2(1-\lambda^2 a^2)E + \lambda^2 a^2 (D+F) - B$

Here $a^2 = 25$, $h=1$ $\therefore \lambda^2 = \frac{1}{a^2} = \frac{1}{25}$

One Period of oscillation $= \frac{2l}{a} = \frac{2(5)}{5} = 2$ sec.

$\therefore$ one Period of oscillation $= 1$ sec. Hence $k = 1/5$ (i.e. Do up to $t=1$)

Put $\lambda = 1/5$ in above formula we get the simplest form

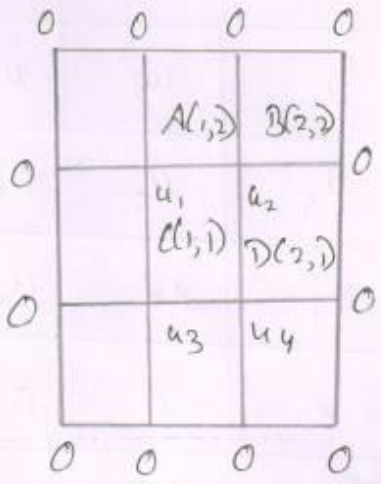
$u_{i,j+1} = (u_{i+1,j} + u_{i-1,j}) - u_{i,j}$ (i.e) $H = (D+F) - B$

Condition $u_t(x,0) = 0$ is equivalent to $E = \frac{(A+C)}{2}$

t \ x	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	-1	-1	-1	-1	0
4/5	0	-2	-3	-3	-2	0
1	0	-2	-4	-4	-2	0

9) Solve the Poisson's Equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with sides $x=0$, $y=0$, $x=3$ and $y=3$ with $u=0$ on the boundary and mesh length is 1.

Hints:



Here $h=1$ and $f(x,y) = -10(x^2+y^2+10)$

Since $f(x,y)$ is symmetrical about the line $x=y$, we get $u_1 = u_4$

∴ It is enough to find u_1, u_2 and u_3

Put $i = 1$ and $j = 2$ in above ~~found~~ formula we get, $-4u_1 + u_2 + u_3 = -150$ ..... (1)

Put $i = 2$ and $j = 2$ in above formula we get, $u_1 - 4u_2 + u_4 = -180$ ..... (2)

Put $i = 1$ and $j = 1$ in above formula we get, $u_1 - 4u_3 + u_4 = -120$ ..... (3)

Put $i = 2$ and $j = 1$ in above formula we get, $u_2 + u_3 - 4u_4 = -150$ ..... (4)

Solving (1), (2), (3) and (4) we get $u_1 = 75 = u_4$, $u_2 = 82.5$ and $u_3 = 67.5$

10) Solve $\nabla^2 u = 8x^2y^2$ over the square with $x = -2, x = 2, y = -2, y = 2$ with $u = 0$ on the boundary and mesh length 1.

Hints:

0	u_1	u_2	u_3	0
0	u_4	u_5	u_6	0
0	u_7	u_8	u_9	0
0	0	0	0	0

Here $h = 1$ and $f(x,y) = 8x^2y^2$

Since $f(x,y)$ is symmetrical about x and y axes and also about the line $x = y$, we get

$u_1 = u_3 = u_7 = u_9$ & $u_2 = u_4 = u_6 = u_8$

∴ It is enough to find u_1, u_2 and u_5

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) = 8i^2j^2$$

Put $i = 1$ and $j = -1$ in above formula we get, $u_2 - 2u_1 = 4$ ... (1)

Put $i = 0$ and $j = 1$ in above formula we get, $2u_1 - 4u_2 + u_5 = 0$ ... (2)

Put $i = 0$ and $j = 0$ in above formula we get $u_2 - u_5 = 0$ ... (3)

Solving (1), (2) and (3) we get $u_1 = -3$, $u_2 = -2$ and $u_5 = -2$

$u_1 = u_3 = u_7 = u_9 = -3$ & $u_2 = u_4 = u_6 = u_8 = -2$ & $u_5 = -2$