QUESTION BANK

DEPARTMENT: CIVIL **SEMESTER: VI**

SUBJECT CODE / Name: CE6602 / STRUCTURAL ANALYSIS-II

UNIT 2 - STIFFNESS MATRIX METHOD

PART - A (2 marks)

1. **Define static indeterminacy.** **(AUC Apr/May 2011)**

The excess number of reactions that make a structure indeterminate is called static indeterminacy.

Static indeterminacy = No. of reactions – Equilibrium conditions

2. **Define flexibility of a structure.** **(AUC Apr/May 2011)**

This method is also called the force method in which the forces in the structure are treated as unknowns. The no of equations involved is equal to the degree of static indeterminacy of the structure.

3. **Write down the equation of element stiffness matrix as applied to 2D plane element.** **(AUC Nov/Dec 2011)**

The equation of element stiffness matrix for 2D plane element is

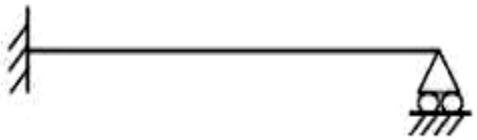
$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

4. **Define degree of freedom of the structure with an example.** **(AUC May/June 2012)**
What is degree of kinematic indeterminacy and give an example. **(AUC Nov/Dec 2011)**

Degree of freedom is defined as the least no of independent displacements required to define the deformed shape of a structure.

There are two types of DOF: (a) Nodal type DOF and (b) Joint type DOF.

For example:



i = r – e where, r = no of reactions, e = no of equilibrium conditions r = 4 and e = 3

i = 4 – 3 = **1**

5. Write a short note on global stiffness matrices. (AUC May/June 2012)

The size of the global stiffness matrix (GSM) = No: of nodes x Degrees of freedom per node.

6. Write a note on element stiffness matrix. (AUC May/June 2013)

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}$$

The element stiffness is K_1, K_2, K_3 etc.....

7. List out the properties of rotation matrix. (AUC May/June 2013)

- Matrix multiplication has no effect on the zero vectors (the coordinates of the origin).
- It can be used to describe rotations about the origin of the coordinate system.
- Rotation matrices provide an algebraic description of such rotations.
- They are used extensively for computations.
- Rotation matrices are square matrices with real entries.

8. What are the basic unknowns in stiffness matrix method?

In the stiffness matrix method nodal displacements are treated as the basic unknowns for the solution of indeterminate structures.

9. Define stiffness coefficient 'kij'.

Stiffness coefficient 'kij' is defined as the force developed at joint 'i' due to unit displacement at joint 'j' while all other joints are fixed.

10. What is the basic aim of the stiffness method?

The aim of the stiffness method is to evaluate the values of generalized coordinates 'r' knowing the structure stiffness matrix 'k' and nodal loads 'R' through the structure equilibrium equation.

{R} = [K] {r}

11. What is the displacement transformation matrix?

The connectivity matrix which relates the internal displacement 'q' and the external displacement 'r' is known as the displacement transformation matrix 'a'.

{q} = [a] {r}

12. How are the basic equations of stiffness matrix obtained?

The basic equations of stiffness matrix are obtained as:

- Equilibrium forces
- Compatibility of displacements
- Force displacement relationships

13. What is meant by generalized coordinates?

For specifying a configuration of a system, a certain minimum no of independent coordinates are necessary. The least no of independent coordinates that are needed to specify the configuration is known as generalized coordinates.

14. Write about the force displacement relationship.

The relationship of each element must satisfy the stress-strain relationship of the element material.

15. Compare flexibility method and stiffness method.

Flexibility matrix method:

- The redundant forces are treated as basic unknowns.
- The number of equations involved is equal to the degree of static indeterminacy of the structure.
- The method is the generalization of consistent deformation method.
- Different procedures are used for determinate and indeterminate structures

Stiffness matrix method:

- The joint displacements are treated as basic unknowns
- The number of displacements involved is equal to the no of degrees of freedom of the structure
- The method is the generalization of the slope deflection method.
- The same procedure is used for both determinate and indeterminate structures.

16. Is it possible to develop the flexibility matrix for an unstable structure?

In order to develop the flexibility matrix for a structure, it has to be stable and determinate.

17. What is the relation between flexibility and stiffness matrix?

The element stiffness matrix ‘k’ is the inverse of the element flexibility matrix ‘f’ and is given by f = 1/k or k = 1/f.

18. List the properties of the stiffness matrix.

- The properties of the stiffness matrix are:
- It is a symmetric matrix
- The sum of elements in any column must be equal to zero.
- It is an unstable element therefore the determinant is equal to zero.

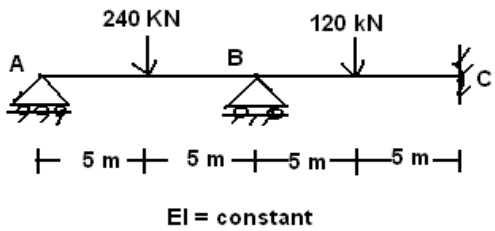
19. Why the stiffness matrix method is also called equilibrium method or displacement method?

Stiffness method is based on the superposition of displacements and hence is also known as the displacement method. And since it leads to the equilibrium equations the method is also known as equilibrium method.

PART - B (16 marks)

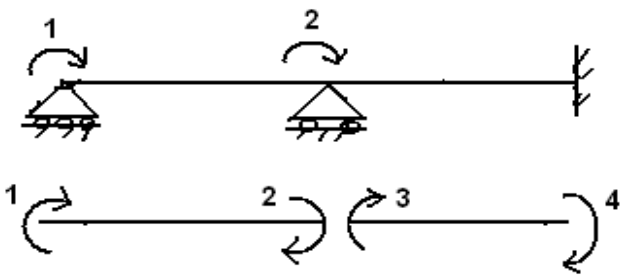
1. Analyse the continuous beam shown in figure using displacement method.

(AUC Apr/May 2011)



Solution:

Step1: Assign coordinates :



Step 2: Fixed End Moment :

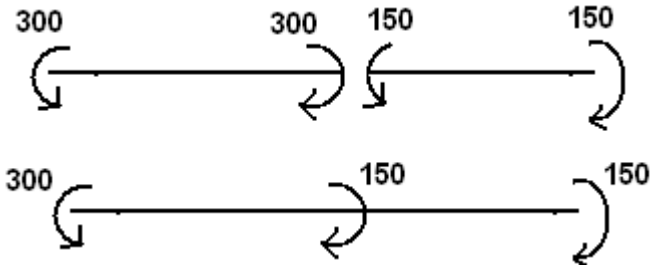
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{240 \text{ x } 10}{8} = -300 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{240 \text{ x } 10}{8} = 300 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{120 \text{ x } 10}{8} = -150 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{120 \text{ x } 10}{8} = 150 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -300 \\ 150 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -300 \\ 150 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0 & 0.4 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

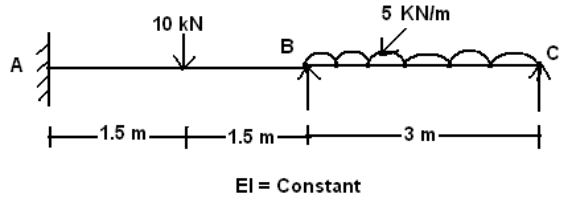
$$P = \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

Step 9: Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -300 \\ 300 \\ -150 \\ 150 \end{bmatrix} + \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

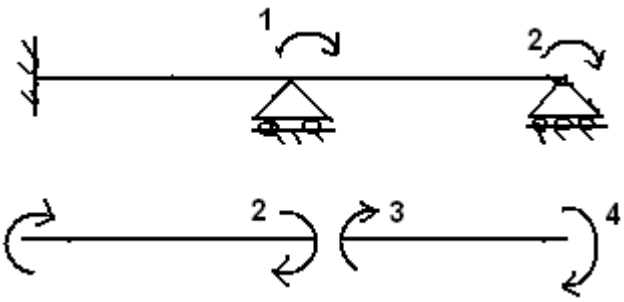
$$M = \begin{bmatrix} 0 \\ 321.9 \\ -321 \\ 64.5 \end{bmatrix}$$

2. Analyse the continuous beam ABC shown in figure by stiffness method and also draw the shear force diagram. (AUC Nov/Dec 2011).



Solution:

Step1: Assign coordinates:



Step 2: Fixed End Moment :

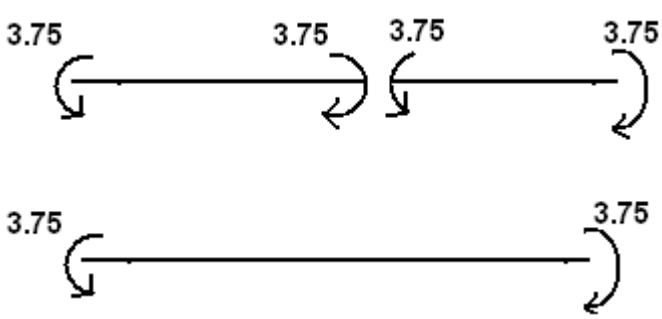
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 3}{8} = -3.75 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 3}{8} = 3.75 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{5 \text{ x } 3^2}{12} = -3.75 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{5 \text{ x } 3^2}{12} = 3.75 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} 0 \\ 3.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.67 & 1.33 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 2.66 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 3.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

Step 8 : Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

$$= \begin{bmatrix} 0.67 & 0 \\ 1.33 & 0 \\ 1.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

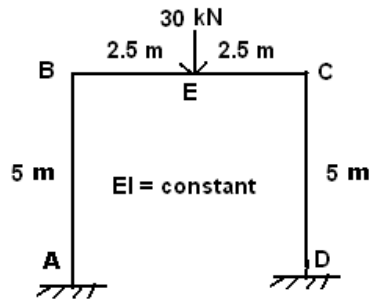
$$P = \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

Step 9 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.75 \\ 3.75 \\ -3.75 \\ 3.75 \end{bmatrix} + \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

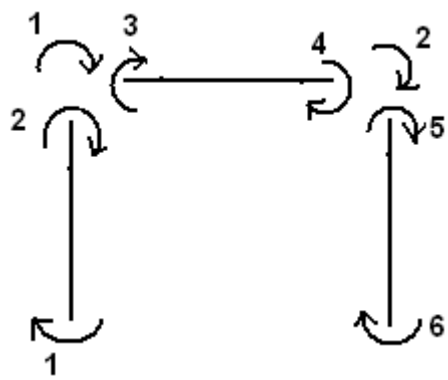
$$M = \begin{bmatrix} -3.205 \\ 4.832 \\ -4.832 \\ 0 \end{bmatrix}$$

3. Analyse the portal frame ABCD shown in figure by stiffness method and also draw the bending moment diagram. (AUC Nov/Dec 2011)



Solution:

Step1: Assign coordinates :



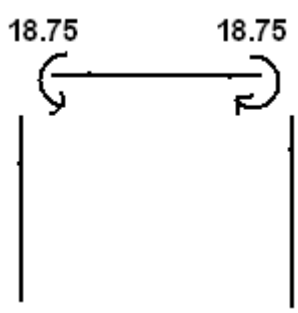
Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{30 \text{ x } 5}{8} = -18.75 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell}{8} = \frac{30 \text{ x } 5}{8} = 18.75 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -18.75 \\ 18.75 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.4 & 0.8 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 1.6 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -18.75 \\ 18.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

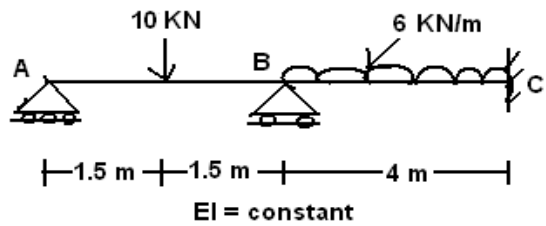
$$= \begin{bmatrix} 0.4 & 0 \\ 0.8 & 0 \\ 0.8 & 0.4 \\ 0.4 & 0.8 \\ 0 & 0.8 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

$$P = \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix}$$

Step 9: Final Moments (M):

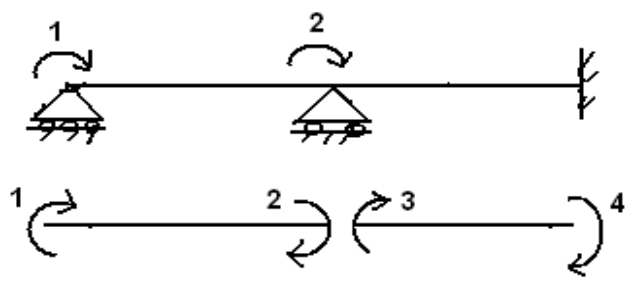
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -18.75 \\ 18.75 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix} = \begin{bmatrix} 6.3 \\ 12.6 \\ -12.5 \\ 12.5 \\ -12.6 \\ -6.3 \end{bmatrix}$$

4. Analyse the continuous beam ABC shown in figure by stiffness method and also sketch the bending moment diagram. (AUC May/June 2012)



Solution:

Step1: Assign coordinates :



Step 2: Fixed End Moment :

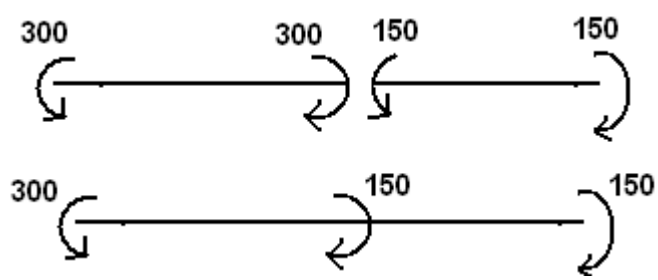
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 3}{8} = -3.75 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 3}{8} = 3.75 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{6 \text{ x } 4^2}{12} = -8 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{6 \text{ x } 4^2}{12} = 8 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -3.75 \\ -4.25 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 2.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 0.879 & -0.253 \\ -0.253 & 0.502 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.879 & -0.253 \\ -0.253 & 0.502 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -3.75 \\ -4.25 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

$$= \begin{bmatrix} 1.33 & 0.67 \\ 0.67 & 1.33 \\ 0 & 1 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 2.221 \\ 1.185 \end{bmatrix}$$

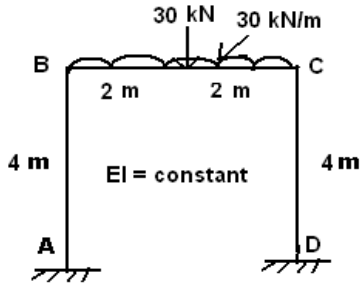
$$P = \begin{bmatrix} 3.75 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

Step 9: Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3.75 \\ 3.75 \\ -8 \\ 8 \end{bmatrix} + \begin{bmatrix} 3.75 \\ 3.06 \\ 1.185 \\ 0.59 \end{bmatrix}$$

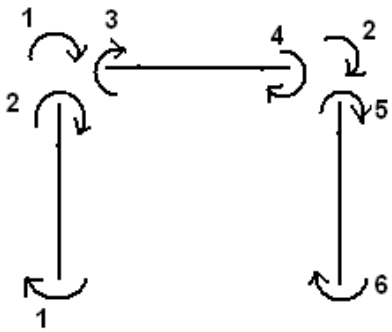
$$M = \begin{bmatrix} 0 \\ 6.81 \\ -6.81 \\ 8.59 \end{bmatrix}$$

5. Analyse the portal frame ABCD shown in figure by stiffness method and also sketch the bending moment diagram. (AUC May/June 2012)



Solution:

Step1: Assign coordinates :



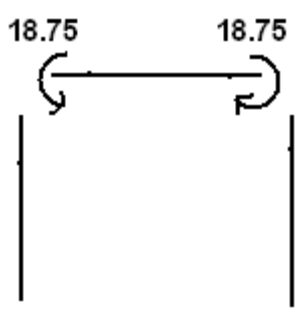
Step 2: Fixed End Moment :

$$M_{FBC} = -\left[\frac{w\ell}{8} + \frac{w\ell^2}{12}\right] = -\left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = -55 \text{ kNm}$$

$$M_{FCB} = \left[\frac{w\ell}{8} + \frac{w\ell^2}{12}\right] = \left[\frac{30 \text{ x } 4}{8} + \frac{30 \text{ x } 4^2}{12}\right] = 55 \text{ kNm}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -55 \\ 55 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 1 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -55 \\ 55 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \\ 1 & 0.5 \\ 0.5 & 1 \\ 0 & 1 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

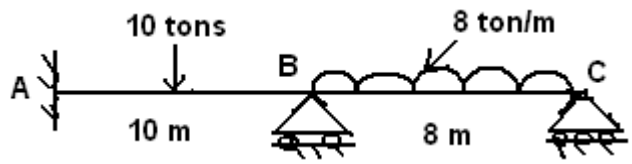
$$P = \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix}$$

Step 9: Final Moments (M):

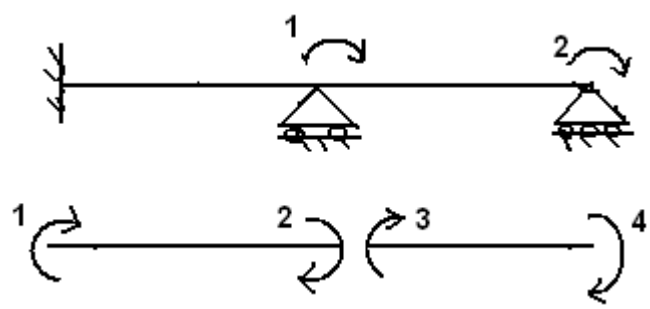
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -55 \\ 55 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix} = \begin{bmatrix} 18.15 \\ 36.3 \\ -36.3 \\ 36.45 \\ -36.3 \\ -18.15 \end{bmatrix}$$

6. A two span continuous beam ABC is fixed at A and simply supported over the supports B and C. AB = 10 m and BC = 8 m. moment of inertia is constant throughout. A single central concentrated load of 10 tons acts on AB and a uniformly distributed load of 8 ton/m acts over BC. Analyse the beam by stiffness matrix method. (AUC May/June 2013)

Solution:



Step1: Assign coordinates :



Step 2: Fixed End Moment :

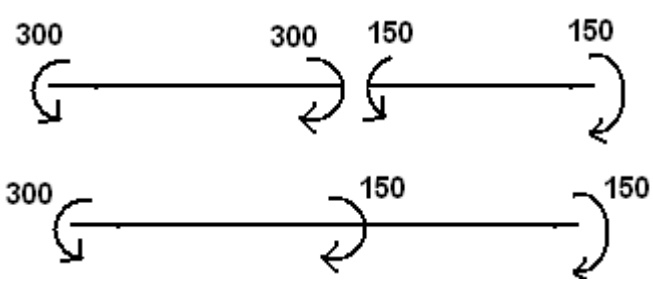
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{10 \text{ x } 10}{8} = -12.5 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{10 \text{ x } 10}{8} = 12.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{8 \text{ x } 8^2}{12} = -42.67 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{8 \text{ x } 8^2}{12} = 42.67 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -30.17 \\ 42.67 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.9 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -30.17 \\ 42.67 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0 \\ 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$P = \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

Step 9: Final Moments (M):

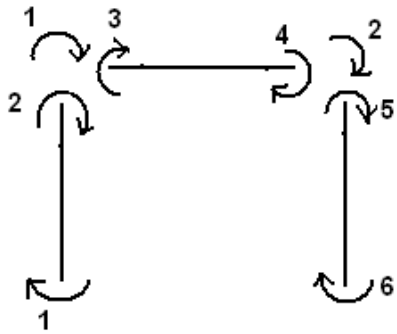
$$M = \mu + P = \begin{bmatrix} -12.5 \\ 12.5 \\ -42.67 \\ 42.67 \end{bmatrix} + \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.83 \\ 39.16 \\ -39 \\ 0 \end{bmatrix}$$

7. A portal frame ABCD with supports A and D are fixed at same level carries a uniformly distributed load of 8 tons/m on the span AB. Span AB = BC = CD = 9 m. EI is constant throughout. Analyse the frame by stiffness matrix method. (AUC May/June 2013)

Solution:

Step1: Assign coordinates :



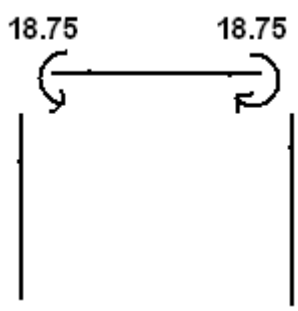
Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{8 \text{ x } 9^2}{12} = -54 \text{ ton.m}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{8 \text{ x } 9^2}{12} = 54 \text{ ton.m}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} -54 \\ 54 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.22 & 0.44 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0.44 & 0.22 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 0.88 & 0.22 \\ 0.22 & 0.88 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -54 \\ 54 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

$$= \begin{bmatrix} 0.22 & 0 \\ 0.44 & 0 \\ 0.44 & 0.22 \\ 0.22 & 0.44 \\ 0 & 0.44 \\ 0 & 0.22 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

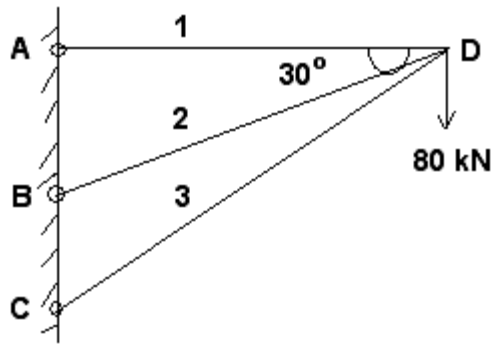
$$P = \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix}$$

Step 9: Final Moments (M):

$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -54 \\ 54 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \\ -36 \\ 36 \\ -36 \\ -18 \end{bmatrix}$$

8. **Using matrix stiffness method, analyze the truss for the member forces in the truss loaded as shown in figure. AE and L are tabulated below for all the three members.**

(AUC Apr/May 2011)

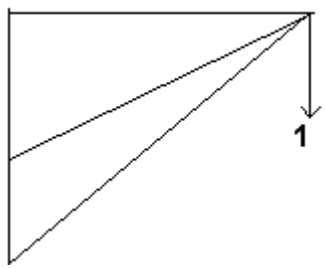


Member	AE	L
AD	400	400
BD	461.9	461.9
CD	800	800

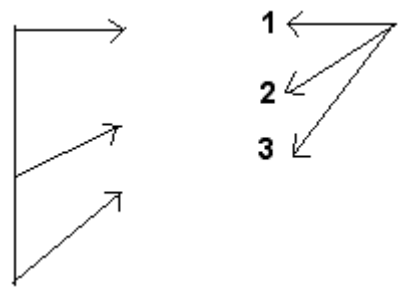
Solution:

Step 1: Assign coordinates:

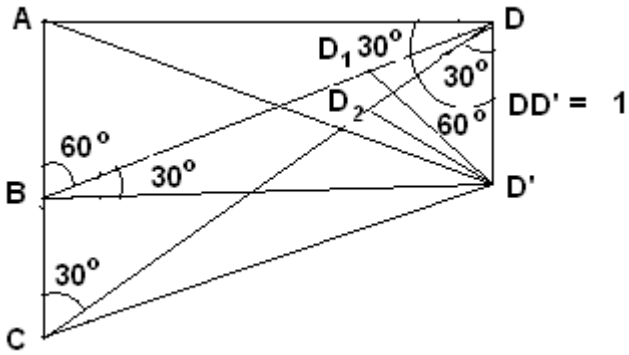
i) Global coordinates:



ii) Local coordinates:



Step 2: Displacement diagram:



Step 3: Formation of [A] matrix:

Apply unit displacement in DD'.

Displacement along 1, AD = 0

Displacement along 2 and 3,

$DD_1 = \cos 60^o = 0.5$ and $DD_2 = \cos 30^o = 0.866$

$$A = \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

Step 4: Stiffness matrix (K):

$$K = \frac{AE}{L}\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5: System stiffness matrix (J):

$$J = A^T K A$$

$$= \begin{matrix} 0 & -0.5 & -0.866 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$= \begin{matrix} 0 & -0.5 & -0.866 \end{matrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$J = 1$$

$$J^{-1} = 1$$

Step 6: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= 1 \text{ x } 80 = 80$$

Step 7: Element forces (P):

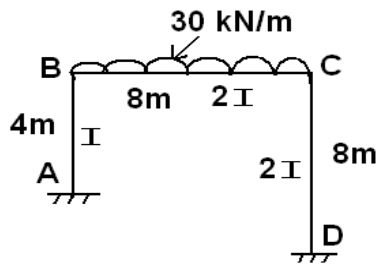
$$P = K A \Delta$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} 80$$

$$= \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} 80$$

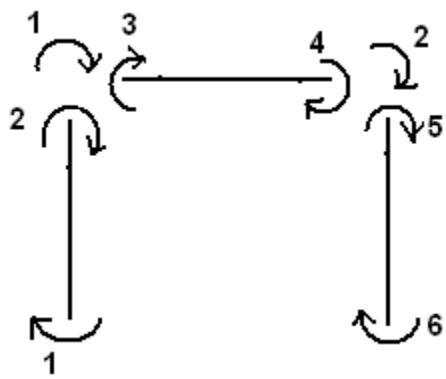
$$\text{Final forces, } P = \begin{bmatrix} 0 \\ -40 \\ -69.28 \end{bmatrix}$$

9. Analyse the frame shown in figure by matrix stiffness method.



Solution:

Step1: Assign coordinates :



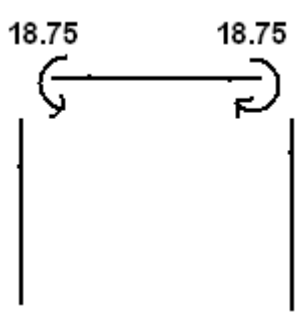
Step 2: Fixed End Moment :

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{30 \text{ x } 8^2}{12} = -160 \text{ kN.m}$$

$$M_{FCB} = \frac{w\ell^2}{12} = \frac{30 \text{ x } 8^2}{12} = 160 \text{ kN.m}$$

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

Step 3: Fixed End Moment Diagram:



$$W^O = \begin{bmatrix} 0 \\ -160 \\ 160 \end{bmatrix}$$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.25 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.125 & 0 & 1 \\ -0.125 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -0.25 & -0.25 & 0 & 0 & -0.125 & -0.125 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

Step 6 : System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{bmatrix} -0.25 & -0.25 & 0 & 0 & -0.125 & -0.125 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}$$

$$= EI \begin{bmatrix} -0.375 & -0.375 & 0 & 0 & -0.187 & -0.187 \\ 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 0.234 & -0.375 & -0.187 \\ -0.375 & 2 & 0.5 \\ -0.187 & 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 6.29 & 1.10 & 0.31 \\ 1.10 & 0.73 & -0.08 \\ 0.31 & -0.08 & 0.55 \end{bmatrix}$$

Step 7: Displacement matrix (Δ):

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{1}{EI}\begin{bmatrix} 6.29 & 1.10 & 0.31 \\ 1.10 & 0.73 & -0.08 \\ 0.31 & -0.08 & 0.55 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -160 \\ 160 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix}$$

Step 8: Element forces (P):

$$P = K\,A\,\Delta$$

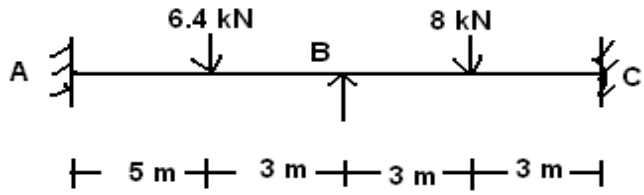
$$= \frac{EI}{EI}\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{8} & 0 & 1 \\ -\frac{1}{8} & 0 & 0 \end{bmatrix}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix}$$

$$P = \begin{bmatrix} -0.375 & 0.5 & 0 \\ -0.375 & 1 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 1 \\ -0.187 & 0 & 1 \\ -0.187 & 0 & 0.5 \end{bmatrix}\begin{bmatrix} 126.4 \\ 129.6 \\ -100.8 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 82.2 \\ 79.2 \\ 36 \\ -124.44 \\ -74.04 \end{bmatrix}$$

Step 9: Final Moments (M):

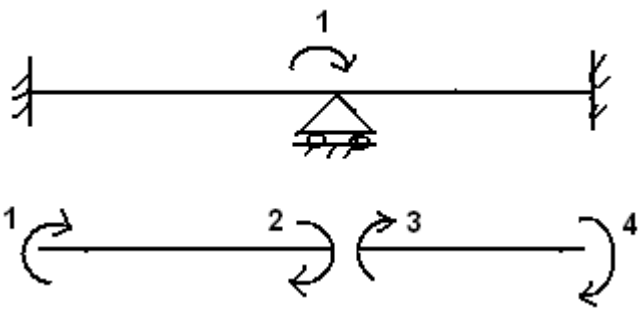
$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -160 \\ 160 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 17.4 \\ 82.2 \\ 79.2 \\ 36 \\ -124.44 \\ -74.04 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 82.2 \\ -81 \\ 124 \\ -124.44 \\ -74.04 \end{bmatrix}$$

10. Analyse the continuous beam shown in figure using displacement method.



Solution:

Step1: Assign coordinates :



Step 2: Fixed End Moment :

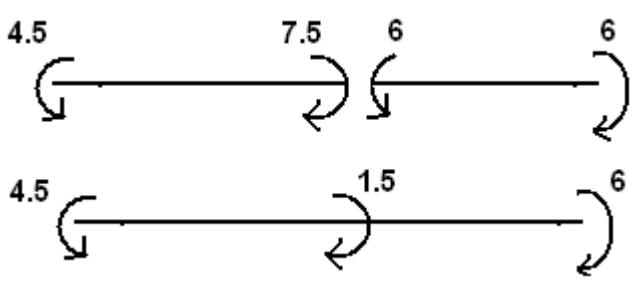
$$M_{FAB} = -\frac{wab^2}{\ell^2} = -\frac{6.4 \text{ x } 5 \text{ x } 3^2}{8^2} = -4.5 \text{ kNm}$$

$$M_{FBA} = \frac{wa^2b}{\ell^2} = -\frac{6.4 \text{ x } 5^2 \text{ x } 3}{8^2} = 7.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell}{8} = -\frac{8 \text{ x } 6}{8} = -6 \text{ kNm}$$

$$M_{FCB} = \frac{w\ell}{8} = \frac{8 \text{ x } 6}{8} = 6 \text{ kNm}$$

Step 3: Fixed End Moment Diagram:



$W^O = 1.5$

Step 4: Formation of (A) matrix:

$$A = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T = \begin{matrix} 0 & 1 & 1 & 0 \end{matrix}$$

Step 5: Stiffness matrix (K):

$$K = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}$$

Step 6: System stiffness matrix (J):

$$J = A^T K A$$

$$= EI \begin{matrix} 0 & 1 & 1 & 0 \end{matrix} \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= EI \begin{matrix} 0.25 & 0.5 & 0.67 & 0.33 \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$J = EI\ 1.17$$

$$J^{-1} = \frac{0.85}{EI}$$

Step 7: Displacement matrix (Δ)

$$\Delta = J^{-1} W$$

$$= J^{-1}\left[W^{*} - W^{0}\right]$$

$$= \frac{0.85}{EI} \quad 0 - 1.5$$

$$\Delta = -\frac{1.275}{EI}$$

Step 8: Element forces (P):

$$P = K\ A\ \Delta$$

$$= \frac{EI}{EI}\begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.67 & 0.33 \\ 0 & 0 & 0.33 & 0.67 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad -1.275$$

$$= \begin{bmatrix} 0.25 \\ 0.5 \\ 0.67 \\ 0.33 \end{bmatrix} \quad -1.275$$

$$P = \begin{bmatrix} -0.319 \\ -0.638 \\ -0.854 \\ -0.421 \end{bmatrix}$$

Step 9: Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -4.5 \\ 7.5 \\ -6 \\ 6 \end{bmatrix} + \begin{bmatrix} -0.319 \\ -0.638 \\ -0.854 \\ -0.421 \end{bmatrix}$$

$$M = \begin{bmatrix} -4.82 \\ 6.86 \\ -6.85 \\ 5.58 \end{bmatrix}$$