



## QUESTION BANK

DEPARTMENT: CIVIL

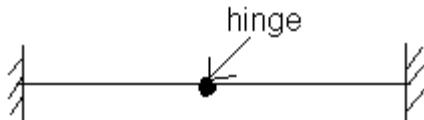
SEMESTER: VI

SUBJECT CODE / Name: CE 6602 / STRUCTURAL ANALYSIS-II

### UNIT 1- FLEXIBILITY METHOD

#### PART - A (2 marks)

1. Find degree of indeterminacy of the following. (AUC Apr/May 2011)



$$\text{Degree of indeterminacy} = \text{No. of reactions} - \text{No. of condition equations}$$

$$= (3 + 2 + 3) - 3 \\ = 5$$

2. Define kinematic redundancy. (AUC Apr/May 2011)

When a structure is subjected to loads, each joint will undergo displacements in the form of translations and rotations. Kinematic redundancy of a structure means the number of unknown joint displacement in a structure.

3. Give the mathematical expression for the degree of static indeterminacy of rigid jointed plane frames. (AUC Nov/Dec 2011)

$$\text{Degree of static indeterminacy} = (\text{No. of closed loops} \times 3) - \text{No. of releases}$$

4. What are the properties which characterize the structure response by means of force-displacement relationship? (AUC Nov/Dec 2011)

- Each element of a flexibility matrix represents a displacement at a coordinate (i) due to a force at a coordinate (j).
- If the matrix of the structure is known, we know the behaviour of the structure.

5. What are the conditions to be satisfied for determinate structures and how are indeterminate structures identified? (AUC May/June 2012)

Determinate structures can be solved using conditions of equilibrium alone ( $H = 0$ ;  $V = 0$ ;  $M = 0$ ). No other conditions are required.

Indeterminate structures cannot be solved using conditions of equilibrium because ( $H \neq 0$ ;  $V \neq 0$ ;  $M \neq 0$ ). Additional conditions are required for solving such structures.

6. Write down the equation for the degree of static indeterminacy of the pin-jointed frames, explaining the notations used. (AUC May/June 2012)

$$\text{Total indeterminacy} = \text{External indeterminacy} + \text{Internal indeterminacy}$$

$$\text{External indeterminacy} = \text{No. of reactions} - \text{No. of equilibrium equations}$$

$$\text{Internal indeterminacy} = m - (2j - 3)$$

**7. Differentiate pin-jointed plane frame and rigid jointed plane frame. (AUC May/June 2013)**

S.No	Pin jointed plane frame	Rigid jointed plane frame
1	The joints permit change of angle between connected members.	The members connected at a rigid joint with maintain the angle between them even under deformation due to loads.
2	The joints are incapable of transferring any moment to the connected members and vice-versa.	Members can transmit both forces and moments between themselves through the joint.
3	The pins transmit forces between connected members by developing shear.	Provision of rigid joints normally increases the redundancy of the structures.

**8. Mention any two methods of determining the joint deflection of a perfect frame.**

(AUC May/June 2013)

- Unit load method
- Virtual work method
- Slope deflection method
- Strain energy method

**9. What are the requirements to be satisfied while analyzing a structure?**

The three conditions to be satisfied are:

- (i) Equilibrium condition
- (ii) Compatibility condition
- (iii) Force displacement condition

**10. What is meant by force method in structural analysis?**

A method in which the forces are treated as unknowns is known as force method.

The following are the force methods:

- Flexibility matrix method
- Consistent deformation method
- Claypeyron's 3 moment method
- Column analogy method

**11. Define flexibility coefficient.**

It is defined as the displacement at coordinate i due to unit force at coordinate j in a structure. It makeup the elements of a flexibility matrix.

**12. Why is flexibility method also called as compatibility method or force method?**

Flexibility method begins with the superposition of forces and is hence known as force method. Flexibility method leads to equations of displacement compatibility and is hence known as compatibility method.

**13. Define the Force Transformation Matrix.**

The connectivity matrix which relates the internal forces Q and the external forces R is known as the force transformation matrix. Writing it in a matrix form,

$$\{Q\} = [b] \{R\}$$

Where, Q = member force matrix/vector; b = force transformation matrix

R = external force/load matrix/ vector

**14. State any two methods of matrix inversion.**

- Adjoint method
- The gauss-jordan method (by linear transformation)
- The Choleski method (by factorization)
- Partitioning method

**15. Define Degree of Freedom and explain its types.**

Degree of freedom is defined as the least no of independent displacements required to define the deformed shape of a structure.

There are two types of DOF: (a) Nodal type DOF and (b) Joint type DOF.

**a) Nodal type DOF:**

This includes the DOF at the point of application of concentrated load or moment, at a section where moment of inertia changes, hinge support, roller support and junction of two or more members.

**b) Joint type DOF:**

This includes the DOF at the point where moment of inertia changes, hinge and roller support and junction of two or more members.

**16. Define a primary structure.**

A structure formed by the removing the excess or redundant restraints from an indeterminate structure making it statically determinate is called primary structure. This is required for solving indeterminate structures by flexibility matrix method.

**17. Briefly mention the two types of matrix methods of analysis of indeterminate structures.**

**Flexibility matrix method:**

This method is also called the force method in which the forces in the structure are treated as unknowns. The no of equations involved is equal to the degree of static indeterminacy of the structure.

**Stiffness matrix method:**

This is also called the displacement method in which the displacements that occur in the structure are treated as unknowns. The no of displacements involved is equal to the no of degrees of freedom of the structure.

**19. Define local and global coordinates.**

**Local coordinates:**

Coordinates defined along the individual member axes locally.

**Global coordinates:**

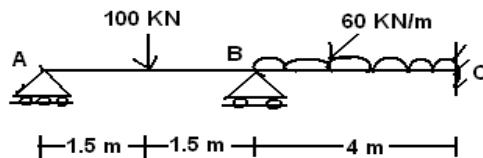
Common coordinate system dealing with the entire structure. Also known as system coordinates.

**20. What is the relation between the flexibility matrix and stiffness matrix?**

The relation between the flexibility matrix and stiffness matrix is that, one is the inverse of the other, when they both exist.

## PART - B (16 marks)

1. Analyse the continuous beam shown in figure using force method. (AUC Apr/May 2011)



**Solution:**

Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (1 + 1 + 3) - 3 = 2$$

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

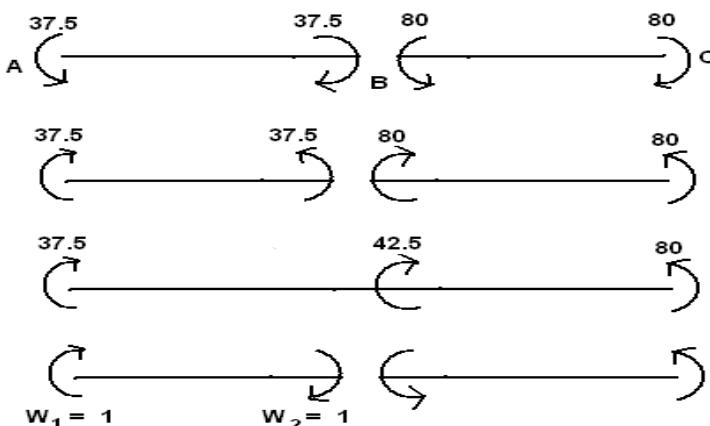
$$M_{FAB} = -\frac{w\ell}{8} = -\frac{100 \times 3}{8} = -37.5 \text{ kNm}$$

$$M_{FBA} = \frac{w\ell}{8} = \frac{100 \times 3}{8} = 37.5 \text{ kNm}$$

$$M_{FBC} = -\frac{w\ell^2}{12} = -\frac{60 \times 4^2}{12} = -80 \text{ kNm}$$

$$M_{FBC} = \frac{w\ell^2}{12} = \frac{60 \times 4^2}{12} = 80 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w B_x$$

$$B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 2.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix}$$

$$\begin{aligned}
F_w &= B_x^T F B_w \\
&= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
F_w &= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Step 6 : Displacement matrix (X):

$$\begin{aligned}
X &= -F_x^{-1} F_w W \\
&= -\frac{EI}{EI} \begin{bmatrix} 0.502 & 0.253 \\ 0.253 & 0.879 \end{bmatrix} \begin{bmatrix} 0.5 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix} \\
&= -\begin{bmatrix} 0.251 & -0.502 \\ 0.127 & -0.253 \end{bmatrix} \begin{bmatrix} 37.5 \\ 42.5 \end{bmatrix} \\
&= -\begin{bmatrix} -11.923 \\ -5.99 \end{bmatrix} \\
X &= \begin{bmatrix} 11.923 \\ 5.99 \end{bmatrix}
\end{aligned}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 37.5 \\ 42.5 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$P = \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -37.5 \\ 37.5 \\ -80 \\ 80 \end{bmatrix} + \begin{bmatrix} 37.5 \\ 30.58 \\ 11.923 \\ 5.99 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 68.08 \\ -68.08 \\ 95.99 \end{bmatrix}$$

2. Analyse the portal frame ABCD shown in figure using force method. (AUC Apr/May 2011)

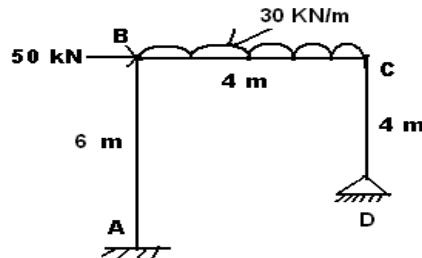


Fig.

**Solution:**

Step1: Static Indeterminacy :

$$\text{Degree of redundancy} = (3 + 2) - 3 = 2$$

Release at B and C by apply hinge.

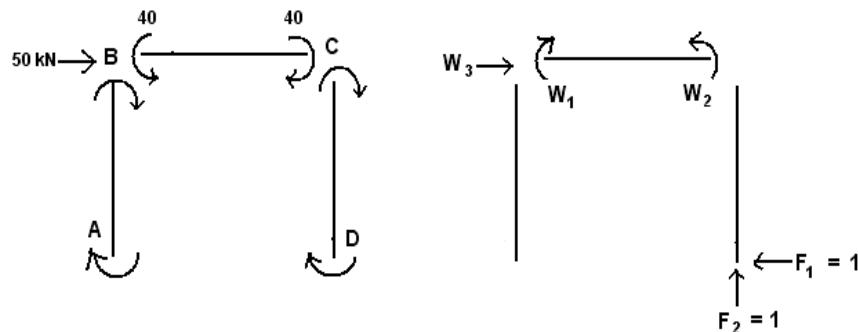
Apply a unit force at B joint.

Step2: Fixed End Moment :

$$M_{FBC} = -\frac{w \ell^2}{12} = -\frac{30 \times 4^2}{12} = -40 \text{ kNm}$$

$$M_{FBC} = \frac{w \ell^2}{12} = \frac{30 \times 4^2}{12} = 40 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 141.28 & -104 \\ -104 & 117.28 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix}$$

$$F_w = B_x^T F B_w$$

$$= \frac{1}{EI} \begin{bmatrix} -2 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -8 & 10 & -8 & 8 & -5.32 & 2.68 \\ 12 & -12 & 5.32 & -2.68 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_w = \frac{1}{EI} \begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix}$$

Step 6 : Displacement matrix (X) :

$$\begin{aligned}
 X &= -F_x^{-1} F_w W \\
 &= -\frac{EI}{EI} \begin{bmatrix} 0.0204 & 0.0181 \\ 0.0181 & 0.0246 \end{bmatrix} \begin{bmatrix} -8 & -5.32 & 48 \\ 5.32 & 0 & -72 \end{bmatrix} \begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix} \\
 &= -\begin{bmatrix} 0.0669 & 0.1085 & 0.3240 \\ 0.0139 & 0.0963 & 0.9024 \end{bmatrix} \begin{bmatrix} 40 \\ -40 \\ 50 \end{bmatrix} \\
 &= -\begin{bmatrix} -14.536 \\ -41.824 \end{bmatrix} \\
 X &= \begin{bmatrix} 14.536 \\ 41.824 \end{bmatrix}
 \end{aligned}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 & -2 & 4 \\ 0 & 0 & 0 & 4 & -4 \\ 1 & 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 40 \\ -40 \\ 50 \\ 14.536 \\ 41.824 \end{bmatrix}$$

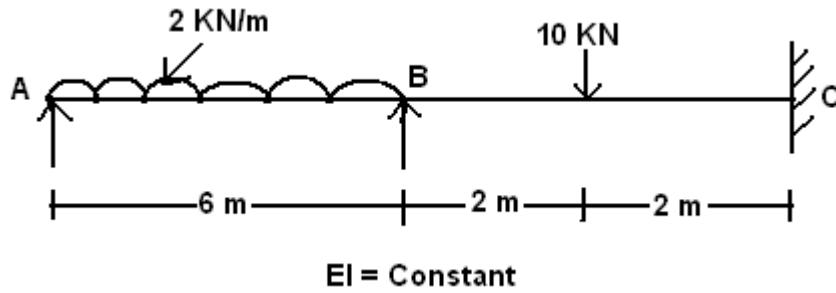
$$P = \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -161.776 \\ -109.152 \\ 149.152 \\ 58.144 \\ -98.144 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -161.776 \\ -109.152 \\ 109.152 \\ 98.144 \\ -98.144 \\ 0 \end{bmatrix}$$

3. Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram. (AUC Nov/Dec 2011).



**Solution:**

Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (1 + 1 + 3) - 3 = 2$$

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

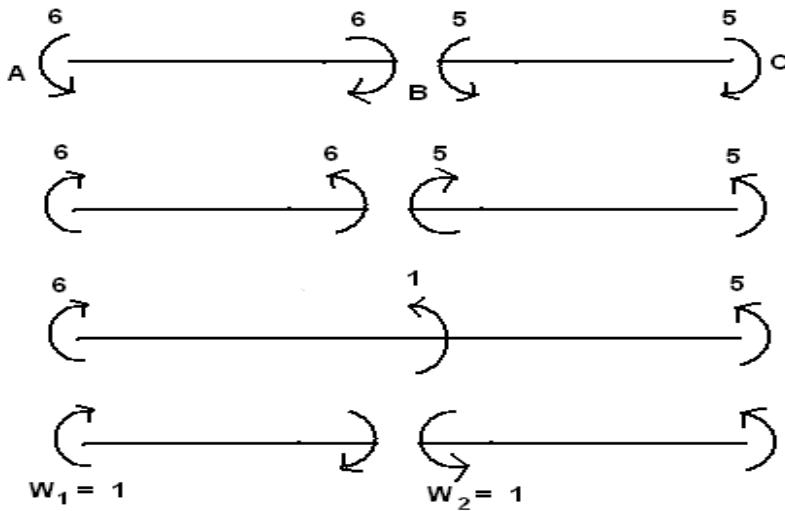
$$M_{FAB} = -\frac{w \ell^2}{12} = -\frac{2 \times 6^2}{12} = -6 \text{ kNm}$$

$$M_{FBA} = \frac{w \ell^2}{12} = \frac{2 \times 6^2}{12} = 6 \text{ kNm}$$

$$M_{FBC} = -\frac{w \ell}{8} = -\frac{10 \times 4}{8} = -5 \text{ kNm}$$

$$M_{FCB} = \frac{w \ell}{8} = \frac{10 \times 4}{8} = 5 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w B_x$$

$$B_w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 3.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix}$$

$$F_w = B_x^T F B_w$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & -2 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_w = \frac{1}{EI} \begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X):

$$X = -F_x^{-1} F_w W$$

$$= -\frac{EI}{EI} \begin{bmatrix} 0.334 & 0.168 \\ 0.168 & 0.837 \end{bmatrix} \begin{bmatrix} 1 & 1.33 \\ 0 & -0.67 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= - \begin{bmatrix} 0.334 & 0.3316 \\ 0.168 & -0.337 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$= - \begin{bmatrix} 1.672 \\ 1.345 \end{bmatrix}$$

$$X = \begin{bmatrix} -1.672 \\ -1.345 \end{bmatrix}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ -1.672 \\ -1.345 \end{bmatrix}$$

$$P = \begin{bmatrix} 6 \\ 1.672 \\ -2.672 \\ -1.345 \end{bmatrix}$$

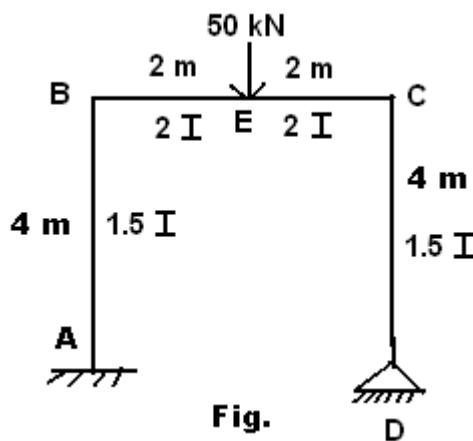
Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -6 \\ 6 \\ -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 6 \\ 1.672 \\ -2.672 \\ -1.345 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 7.672 \\ -7.672 \\ 3.655 \end{bmatrix}$$

4. Analyse the portal frame ABCD shown in figure by flexibility matrix method and sketch the bending moment diagram.

(AUC Nov/Dec 2011)



**Solution:**

Step 1: Static Indeterminacy :

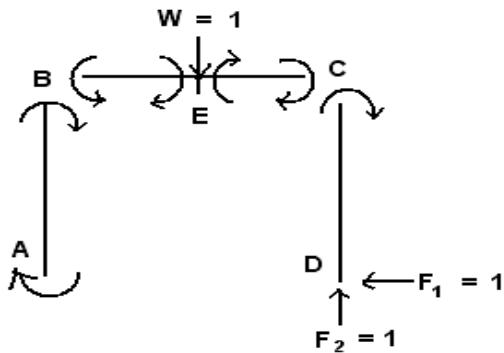
$$\text{Degree of redundancy} = (3 + 2) - 3 = 2$$

Release at D by apply horizontal and vertical supports.

Step 2: Fixed End Moment :

$$M_{FAB} = M_{FBA} = M_{FBC} = M_{FBC} = M_{FCD} = M_{FDC} = 0$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -4 \\ -4 & 4 \\ 4 & -2 \\ -4 & 2 \\ 4 & 0 \\ -4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 60.48 & -37.28 \\ -37.28 & 53.2 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix}$$

$$F_w = B_x^T F B_w$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & 4 & -4 & 4 & -4 & 4 & -4 & 0 \\ 4 & -4 & 4 & -2 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.89 & -0.44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.44 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.33 & -0.17 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.17 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.33 & -0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.17 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.89 & -0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.89 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -1.76 & 3.56 & -2 & 2 & -2 & 2 & -3.56 & 1.76 \\ 5.32 & -5.32 & 1.66 & -1.34 & 0.66 & -0.34 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_w = \frac{1}{EI} \begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix}$$

Step 6 : Displacement matrix (X):

$$X = -F_x^{-1} F_w W$$

$$= -\frac{EI}{EI} \begin{bmatrix} 0.0291 & 0.0203 \\ 0.0203 & 0.033 \end{bmatrix} \begin{bmatrix} 14.64 \\ -24.60 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -0.0734 \\ -0.5146 \end{bmatrix} 50$$

$$= -\begin{bmatrix} -3.67 \\ -25.73 \end{bmatrix}$$

$$X = \begin{bmatrix} 3.67 \\ 25.73 \end{bmatrix}$$

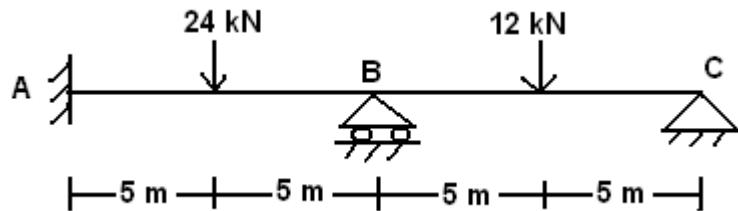
Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} -2 & 0 & 4 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \\ 0 & 4 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 3.67 \\ 25.73 \end{bmatrix}$$

$$P = \begin{bmatrix} 2.92 \\ 11.76 \\ -11.76 \\ -36.78 \\ 36.78 \\ 14.68 \\ -14.68 \\ 0 \end{bmatrix}$$

The final moments also same, since there are no external forces acting on the members.

5. Analyse the continuous beam ABC shown in figure by flexibility matrix method and sketch the bending moment diagram.  
 (AUC May/June 2012)



**Solution:**

Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (3 + 1 + 1) - 3 = 2$$

Release at A and B by apply hinge.

Step 2: Fixed End Moment :

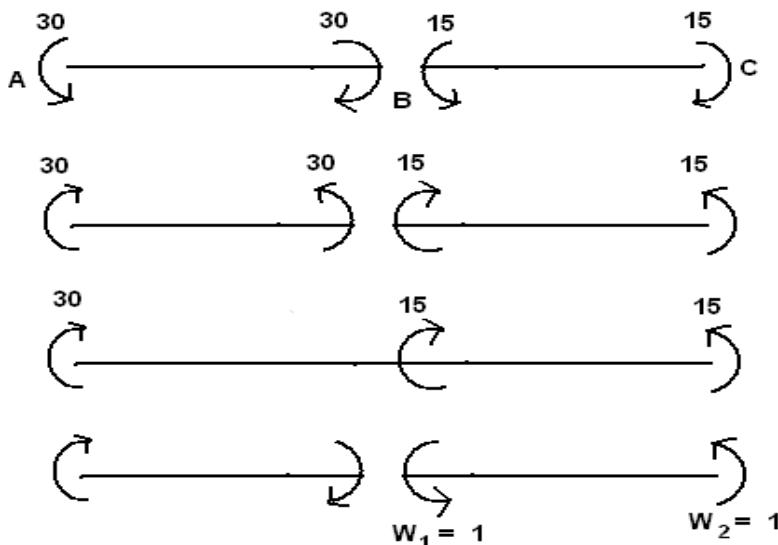
$$M_{FAB} = -\frac{w \ell}{8} = -\frac{24 \times 10}{8} = -30 \text{ kNm}$$

$$M_{FBA} = \frac{w \ell}{8} = \frac{24 \times 10}{8} = 30 \text{ kNm}$$

$$M_{FBC} = -\frac{w \ell}{8} = -\frac{12 \times 10}{8} = -15 \text{ kNm}$$

$$M_{FCB} = \frac{w \ell}{8} = \frac{12 \times 10}{8} = 15 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 3.33 & 1.67 \\ 1.67 & 6.66 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix}$$

$$\begin{aligned}
F_w &= B_x^T F B_w \\
&= \frac{1}{EI} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ -1.67 & 3.33 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \frac{1}{EI} \begin{bmatrix} 3.33 & -1.67 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
F_w &= \frac{1}{EI} \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}
\end{aligned}$$

Step 6 : Displacement matrix (X):

$$\begin{aligned}
X &= -F_x^{-1} F_w W \\
&= -\frac{EI}{EI} \begin{bmatrix} 0.3435 & -0.086 \\ -0.086 & 0.1717 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} -15 \\ -15 \end{bmatrix} \\
&= -\begin{bmatrix} -0.286 & 0.144 \\ 0.144 & -0.286 \end{bmatrix} \begin{bmatrix} -15 \\ -15 \end{bmatrix} \\
&= \begin{bmatrix} 2.13 \\ -4.29 \end{bmatrix} \\
X &= \begin{bmatrix} -2.13 \\ 4.29 \end{bmatrix}
\end{aligned}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -15 \\ -15 \\ -2.13 \\ 4.29 \end{bmatrix}$$

$$P = \begin{bmatrix} -2.13 \\ -4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -30 \\ 30 \\ -15 \\ 15 \end{bmatrix} + \begin{bmatrix} -2.13 \\ -4.29 \\ -10.71 \\ -15 \end{bmatrix}$$

$$M = \begin{bmatrix} -32.13 \\ 25.71 \\ -25.71 \\ 0 \end{bmatrix}$$

6. A cantilever of length 15 m is subjected to a single concentrated load of 50 kN at the middle of the span. Find the deflection at the free end using flexibility matrix method. EI is uniform throughout.

(AUC May/June 2013)

**Solution:**

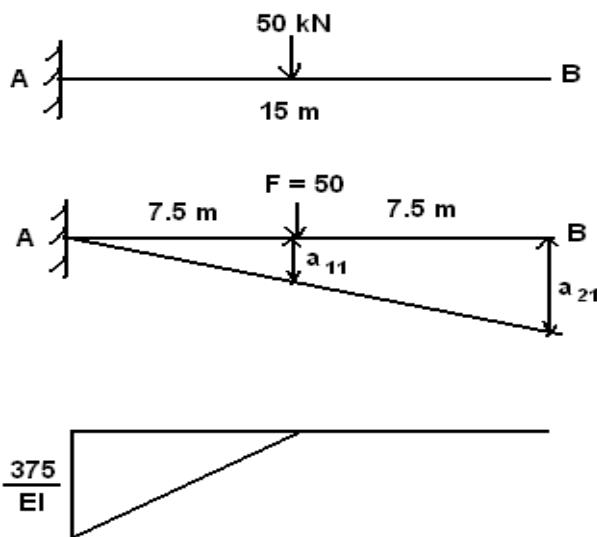
Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = 3 - 3 = 0$$

It is static determinate structures.

Step 2: Deflection at B :

Apply a unit force at given load.



The deflection is calculated by  $\frac{M}{EI}$ .

$$\text{Deflection at } a_{21} = \left( \frac{1}{2} \times 7.5 \times \frac{375}{EI} \right) \times \left( \frac{2 \times 7.5}{3} + 7.5 \right)$$

$$\text{Deflection at B} = \frac{17578.125}{EI}$$

Hint: To find the deflection, we use  $\frac{M}{EI}$  diagram.

7. A two span continuous beam ABC is fixed at A and hinged at support B and C. Span AB = BC = 9m. Set up flexibility influence coefficient matrix assuming vertical reaction at B and C as redundant. (AUC May/June 2013)

**Solution:**



Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (3 + 1 + 1) - 3 = 2$$

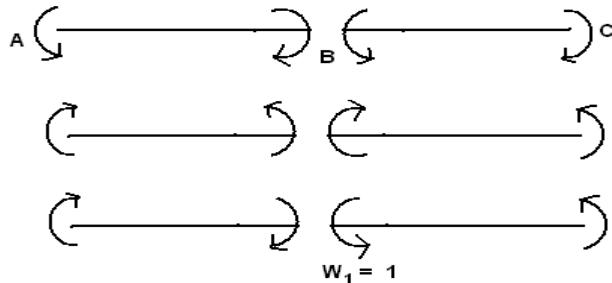
Release at A and B by apply hinge.

Step 2: Fixed End Moment :

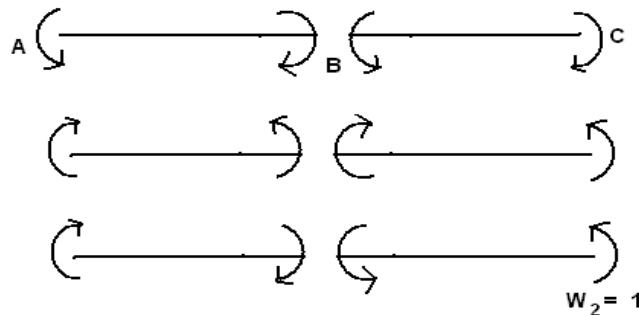
$$M_{FAB} = M_{FBA} = M_{FBC} = M_{FBC} = 0$$

Step 3: Equivalent Joint Load:

Case (i):



Case (ii):



Step 3: Flexibility Influence Co-efficient Matrix (B):

For case (i):

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

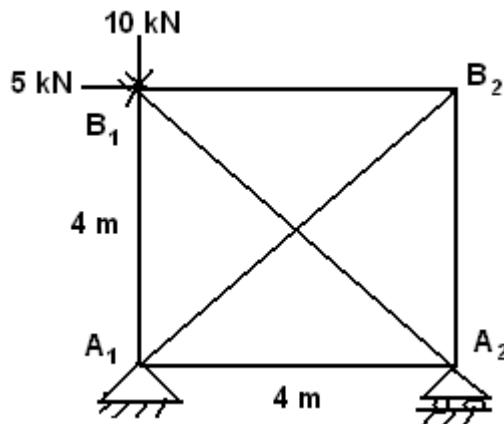
For case (ii):

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

8. A Statically indeterminate frame shown in figure carries a load of 80 kN. Analyse the frame by matrix flexibility method. A and E are same for all members. (AUC May/June 2012)



**Solution:**

Step 1: Static Indeterminacy:

$$\begin{aligned}\text{Degree of redundancy} &= \text{Internal Indeterminate} - \text{External Indeterminate} \\ &= [m - (2j - 3)] - (r - R) \\ &= [6 - (8 - 3)] - (3 - 3) \\ &= 1\end{aligned}$$

Step 2: Member forces:

Take member AD as a redundant.

$$\tan \theta = \frac{3}{4} = 0.75; \sin \theta = 0.6; \cos \theta = 0.8;$$

$$\underline{\Sigma V=0}$$

$$V_A = 1$$

$$\underline{\Sigma M = 0}$$

$$H_A = 1.333 \quad \text{and} \quad H_B = 1.333$$

At joint D:

$$F_{DC} = 1 \text{ (compression)} = -1$$

At joint C:

$$\underline{\Sigma V=0}$$

$$F_{CA} \sin \theta = 1$$

$$F_{CA} = 1.667; F_{CB} = 1.333$$

At joint B:

$$F_{BA} = 0; F_{BC} = 1.333$$

Analyse by method of joints and find the member forces.

Step 3: Flexibility Co-efficient Matrix:

$$B = B_w \ B_x$$

$$B_w = \begin{bmatrix} 0 \\ -1.333 \\ -1 \\ 1.667 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \\ -1.25 \\ -1.25 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0.75 \\ -1.333 & 1 \\ -1 & 0.75 \\ 1.667 & -1.25 \\ 0 & -1.25 \\ 0 & 1 \end{bmatrix}$$

Step 4: Flexibility matrix (F):

$$F = \frac{1}{AE} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \begin{bmatrix} 0.75 & 1 & 0.75 & -1.25 & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \\ -1.25 \\ -1.25 \\ 1 \end{bmatrix}$$

$$F_x = \frac{27}{AE}$$

$$F_x^{-1} = \frac{AE}{27}$$

$$F_w = B_w^T F B_w$$

$$= \begin{bmatrix} 0.75 & 1 & 0.75 & -1.25 & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1.333 \\ -1 \\ 1.667 \\ 0 \\ 0 \end{bmatrix}$$

$$F_w = -\frac{7.30}{AE}$$

Step 5 : Displacement matrix (X) :

$$\begin{aligned} X &= -F_x^{-1} F_w W \\ &= \left[ \frac{-AE}{27} \right] \left[ \frac{-7.30}{AE} \right] 80 \end{aligned}$$

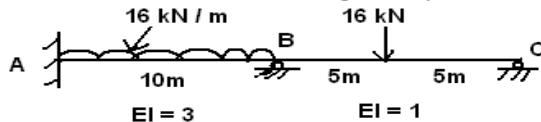
$$X = 21.63 \text{ kN}$$

Step 6 : Internal forces (P) :

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0.75 \\ -1.333 & 1 \\ -1 & 0.75 \\ 1.667 & -1.25 \\ 0 & -1.25 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 80 \\ 21.63 \end{bmatrix}$$

$$\text{Final forces, } P = \begin{bmatrix} 16.22 \\ -84.77 \\ -63.78 \\ 105.76 \\ -27.04 \\ 21.63 \end{bmatrix}$$

### 9. Analyse the continuous beam shown in figure by flexibility method.



**Solution:**

Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (3 + 1 + 1) - 3 = 2$$

Release at A and B by apply hinge.

Step 2: Fixed End Moment :

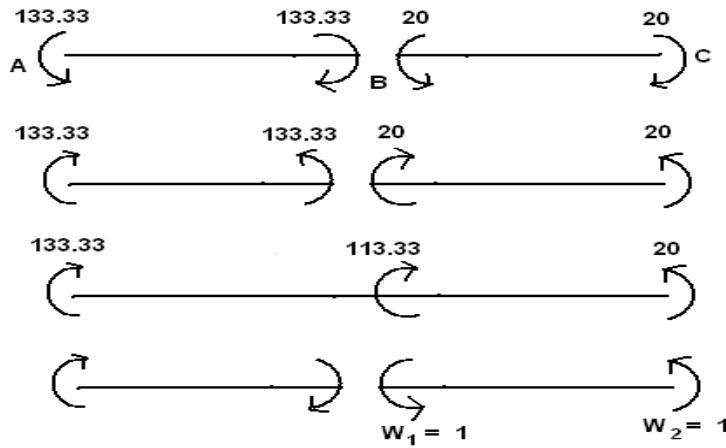
$$M_{FAB} = -\frac{w \ell^2}{12} = -\frac{16 \times 10^2}{12} = -133.33 \text{ kNm}$$

$$M_{FBA} = \frac{w \ell^2}{12} = \frac{16 \times 10^2}{12} = 133.33 \text{ kNm}$$

$$M_{FBC} = -\frac{w \ell}{8} = -\frac{16 \times 10}{8} = -20 \text{ kNm}$$

$$M_{FCB} = \frac{w \ell}{8} = \frac{16 \times 10}{8} = 20 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w B_x$$

$$B_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ 0.56 & -1.11 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \begin{bmatrix} 1.11 & 0.56 \\ 0.56 & 4.44 \end{bmatrix}$$

$$F_x^{-1} = \begin{bmatrix} 0.962 & -0.121 \\ -0.121 & 0.241 \end{bmatrix}$$

$$F_w = B_x^T F B_w$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ -0.56 & 1.11 & 0 & 0 \\ 0 & 0 & 3.33 & -1.67 \\ 0 & 0 & -1.67 & 3.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.11 & -0.56 & 0 & 0 \\ 0.56 & -1.11 & 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_w = \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix}$$

Step 6 : Displacement matrix (X):

$$\begin{aligned} X &= -F_x^{-1} F_w W \\ &= -\begin{bmatrix} 0.962 & -0.121 \\ -0.121 & 0.241 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3.33 & -1.67 \end{bmatrix} \begin{bmatrix} -113.33 \\ -20 \end{bmatrix} \\ &= -\begin{bmatrix} 41.62 \\ -82.90 \end{bmatrix} \\ X &= \begin{bmatrix} -41.62 \\ 82.90 \end{bmatrix} \end{aligned}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -113.33 \\ -20 \\ -41.62 \\ 82.90 \end{bmatrix}$$

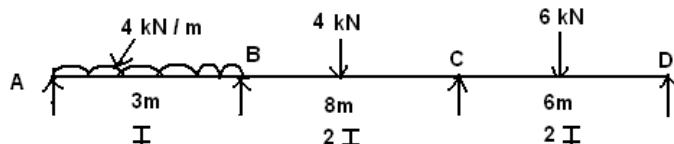
$$P = \begin{bmatrix} -41.62 \\ -82.90 \\ -30.43 \\ -20 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -133.33 \\ 133.33 \\ -20 \\ 20 \end{bmatrix} + \begin{bmatrix} -41.62 \\ -82.90 \\ -30.43 \\ -20 \end{bmatrix}$$

$$M = \begin{bmatrix} -174.95 \\ 50.43 \\ -50.43 \\ 0 \end{bmatrix}$$

**10. Analyse the continuous beam shown in figure by flexibility method.**



**Solution:**

Step 1: Static Indeterminacy :

$$\text{Degree of redundancy} = (1+1+1+1) - 2 = 2$$

Release at B and C by apply hinge.

Step 2: Fixed End Moment :

$$M_{FAB} = -\frac{w \ell^2}{12} = -\frac{4 \times 3^2}{12} = -3 \text{ kNm}$$

$$M_{FBA} = \frac{w \ell^2}{12} = \frac{4 \times 3^2}{12} = 3 \text{ kNm}$$

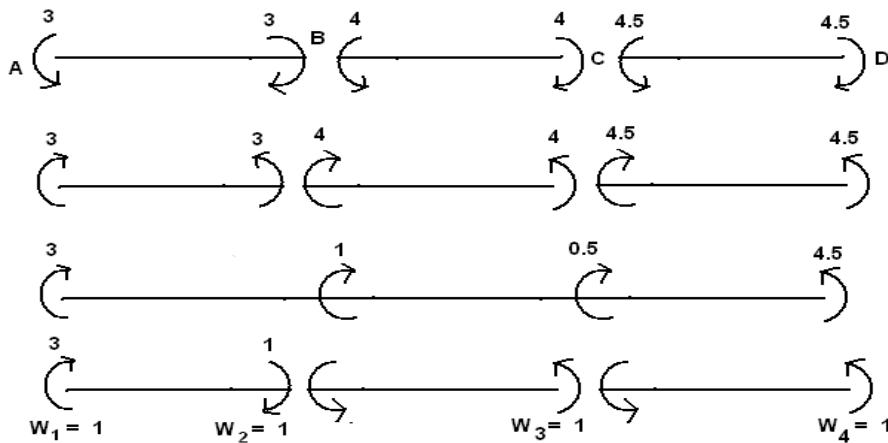
$$M_{FBC} = -\frac{w \ell}{8} = -\frac{4 \times 8}{8} = -4 \text{ kNm}$$

$$M_{FCB} = \frac{w \ell}{8} = \frac{4 \times 8}{8} = 4 \text{ kNm}$$

$$M_{FCD} = -\frac{w \ell}{8} = -\frac{6 \times 6}{8} = -4.5 \text{ kNm}$$

$$M_{FDC} = \frac{w \ell}{8} = \frac{6 \times 6}{8} = 4.5 \text{ kNm}$$

Step 3: Equivalent Joint Load:



Step 4: Flexibility co-efficient matrix (B):

$$B = B_w B_x$$

$$B_w = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B_x = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: Flexibility matrix (F):

$$F = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$F = \frac{1}{EI} \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix}$$

$$F_x = B_x^T F B_x$$

$$= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & 0.67 & -1.33 & 1 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F_x = \frac{1}{EI} \begin{bmatrix} 2.33 & 0.67 \\ 0.67 & 2.33 \end{bmatrix}$$

$$F_x^{-1} = EI \begin{bmatrix} 0.468 & -0.135 \\ -0.135 & 0.468 \end{bmatrix}$$

$$\begin{aligned}
F_w &= B_x^T F B_w \\
&= \frac{1}{EI} \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & -0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & 1.33 & -0.67 & 0 & 0 \\ 0 & 0 & 0.67 & -1.33 & 1 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
F_w &= \frac{1}{EI} \begin{bmatrix} 0.5 & -1 & -0.67 & 0 \\ 0 & 0 & -1.33 & -0.5 \end{bmatrix}
\end{aligned}$$

Step 6 : Displacement matrix (X) :

$$\begin{aligned}
X &= -F_x^{-1} F_w W \\
&= -\frac{EI}{EI} \begin{bmatrix} 0.468 & -0.135 \\ -0.135 & 0.468 \end{bmatrix} \begin{bmatrix} 0.5 & -1 & -0.67 & 0 \\ 0 & 0 & -1.33 & -0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \end{bmatrix} \\
&= -\begin{bmatrix} 0.234 & -0.468 & -0.134 & 0.068 \\ -0.068 & 0.135 & -0.599 & -0.234 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \end{bmatrix} \\
&= -\begin{bmatrix} -0.139 \\ 0.685 \end{bmatrix} \\
X &= \begin{bmatrix} 0.139 \\ -0.685 \end{bmatrix}
\end{aligned}$$

Step 7 : Internal forces (P):

$$P = B \begin{bmatrix} W \\ X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0.5 \\ -4.5 \\ 0.139 \\ -0.685 \end{bmatrix} = \begin{bmatrix} 3 \\ 0.861 \\ 0.139 \\ 1.185 \\ -0.685 \\ -4.5 \end{bmatrix}$$

Step 8 : Final Moments (M):

$$M = \mu + P = \begin{bmatrix} -3 \\ 3 \\ -4 \\ 4 \\ -4.5 \\ 4.5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0.861 \\ 0.139 \\ 1.185 \\ -0.685 \\ -4.5 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 3.861 \\ -3.861 \\ 5.185 \\ -5.185 \\ 0 \end{bmatrix}$$