**QUESTION WITH ANSWERS**

**DEPARTMENT : CIVIL** **SEMESTER:I V**

**SUB.CODE/ NAME: CE8402 / Strength of Materials - II**

## UNIT– 4 STATE OF STRESS IN THREE DIMESIONS

## PART - A (2 marks)

**1. Define stress**

When a certain system of external forces acts on a body then the body offers resistance to these forces. This internal resistance offered by the body per unit area is called the stress induced in the body.

**2. Define principal planes and principal stress.(AUC Nov/Dec 2013) (AUC Apr/May 2010) (AUC Nov/Dec 2010) (AUC Apr/May 2011)**

The plane in which the shear stress is zero is called principal planes. The plane which is independent of shear stress is known as principal plane.

The normal stress acting on prinicipal planes is called principal stress

**3. Define spherical tensor.**

$$\tau_{ij}^{ii} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

It is also known as hydrostatic stress tensor

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$\sigma_m$ is the mean stress.

**4. Define Deviator stress tensor**

$$\tau_{ij}^{1} = \begin{bmatrix} \sigma_x - \sigma_m & l_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & l_{yz} & \sigma_z - \sigma_m \end{bmatrix}$$

**5. Define volumetric strain.** **(AUC Nov/Dec 2010)**

It is defined as the ratio between change in volume and original volume of the body and is denoted by $e_v$

$$e_v = \frac{\text{Change in volume}}{\text{Original volume}} \quad \frac{\delta v}{v}$$

**6. State the principal theories of failure.**

1. Maximum principal stress theory
2. Maximum shear stress (or) stress difference theory
3. Strain energy theory
4. Shear strain energy theory
5. Maximum principal strain theory
6. Mohr's Theory

**7. State the Limitations of Maximum principal stress theory**

1. On a mild steel specimen when spiel tension test is carried out sliding occurs approximately $45^o$ to the axis of the specimen; this shows that the failure in this case is due to maximum shear stress rather than the direct tensile stress.

2. It has been found that a material which is even though weak in simple compression yet can sustain hydrostatic pressure for in excess of the elastic limit in simple compression.

**8. Explain maximum principal stress theory.** **(AUC Nov/Dec 2011)**

According to this theory failure will occur when the maximum principle tensile stress ($\sigma_1$) in the complex system reaches the value of the maximum stress at the elastic limit ($\sigma_{et}$) in the simple tension.

**9. Define maximum shear stress theory**

This theory implies that failure will occur when the maximum shear stress $\tau_{maximum}$ in the complex system reaches the value of the maximum shear stress in simple tension at elastic limit (i.e)

$$l_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{et}}{2} \qquad \text{(or)} \quad \sigma_1 - \sigma_3 = \sigma_{et}$$

**10. State the limitations of maximum shear stress theory.**

i. The theory does not give accurate results for the state of stress of pure shear in which the maximum amount of shear is developed (i.e) Torsion test.

**ii.** The theory does not give us close results as found by experiments on ductile materials. However, it gives safe results.

**11. Explain shear strain Energy theory.**

This theory is also called " Distortion energy Theory" or "Von Mises - Henky Theory."

According to this theory the elastic failure occurs where the shear strain energy per unit volume in the stressed material reaches a value equal to the shear strain energy per unit volume at the elastic limit point in the simple tension test.

**12. State the limitations of Distortion energy theory.**

1. The theory does to agree the experiment results for the material for which $\sigma at$ is quite different etc.
2. This theory is regarded as one to which conform most of the ductile material under the action of various types of loading.

**13. Explain Maximum principal strain theory**

The theory states that the failure of a material occurs when the principal tensile strain in the material reaches the strain at the elastic limit in simple tension (or) when the min minimum principal strain (ie ) maximum principal compressive strain reaches the elastic limit in simple compression.

**14. State the Limitations in maximum principal strain theory**

i. The theory overestimates the behaviour of ductile materials.
ii. The theory does no fit well with the experimental results except for brittle materials for biaxial tension.

**15. State the stress tensor in Cartesian components**

$$\tau_{ij}^{'} = \begin{bmatrix} \sigma_x . & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

**16. Explain the three stress invariants. (AUC Nov/Dec 2013) (AUC May/June 2012)**

The principal stresses are the roots of the cubic equation,

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau^2 xy - \tau y^2{}_z - \tau^2 xz$$

$$I_3 = \sigma_x \sigma_y \sigma_Z - \sigma_x\tau^2{}_{xy} - \sigma_y\tau^2{}_{xz} - \sigma_z\tau^2{}_{xy} + 2\tau_{xy}\tau_{yz}\tau_{xz}$$

**17. State the two types of strain energy**

i. Strain energy of distortion (shear strain energy)
ii. Strain energy of dilatation.

**18. Explain Mohr's Theory**

Let $\tau = f(\sigma)$

The enveloping curve $\tau = f(\sigma)$ must represent in this abscissa σ and ordinates e, the normal and shearing stresses in the plane of slip.

$$\left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2$$

Let $P = \frac{1}{2}(\sigma_1 + \sigma_3)$

$$\tau m = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$(\sigma - p)^2 + \tau^2 = lm^2$$

**19. State the total strain energy theory.**

The total strain energy of deformation is given by

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

and strain energy in simple tension is

$$U = \frac{\sigma_0^2}{2E}$$

**20. State the shear strain energy per unit volume**

$$\sigma_s = \frac{1}{12C}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

where $C = \frac{E}{2\left(1 + \frac{1}{m}\right)}$

**21 Explain the concept of stress?**

When certain system of external forces act on a body then the body offers resistance to these forces. This internal resistance offered by the body per unit area is called the stress induced in the body.

The stress $\sigma$ may be resolved into two components. The first one is the normal stress $\sigma_n$, which is the perpendicular to the section under examination and the second one is the shear stress $\tau$ , which is operating in the plane of the section.

**22. State the Theories of failure.**

**The principal theories are:**

1. Maximum principal stress theory
2. Maximum shear stress (or) stress difference theory
3. Strain energy theory
4. Shear strain energy theory
5. Maximum principal strain theory
6. Mohr's Theory

**23. Define factor of safety .** **(AUC Nov/Dec 2012)**

The ratio of ultimate stress to the working stress is known as factor of safety. However case of elastic material , it is taken as the ratio of yield stress or 0.2% proof stress to working stress .

## PART - B (16 marks)

1. **For the state of stress shown in fig. find the principal plane and principal stress.**

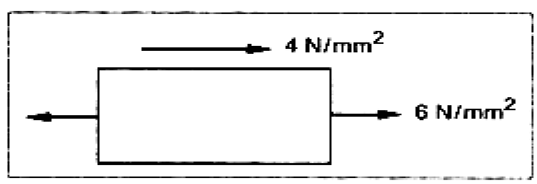



*Fig. 4.7.*

***Solution:***

***Given Data:***

$$\sigma_x = 6\ \text{N/mm}^2$$

$$\tau_{xy} = 4\ \text{N/mm}^2$$

$$\sigma_y = 0$$

$$\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{6}{2} + \sqrt{\left(\frac{6}{2}\right)^2 + 4^2}$$

$$\sigma_1 = 8\ \text{N/mm}^2 \quad \textbf{Ans.}$$

$$\sigma_2 = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{6}{2} - \sqrt{\left(\frac{6}{2}\right)^2 + 4^2}$$

$$= \mathbf{-2\ N/mm^2}$$

$$\sigma_2 = 2\ \text{N/mm}^2\ \text{(compressive)}$$

$$\tan 2\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 4}{6} = 1.33$$

Since $\theta$ is general angle, the specific angles representing the principal planes are designated as $\phi_1$ and $\phi_2$ .

$$\therefore\ 2\,\phi = 53°\,3' \text{ and } 126°\,56'$$

$$\text{Using } 2\,\phi_1 = 53°\,3'$$

$$\sigma_n = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\,\phi_1 + \tau\ \sin 2\,\phi_1$$

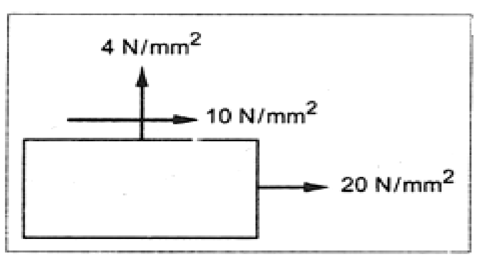
$$= 3 + 3\cos 53°\,3' + 4\sin 53°\,3'$$

$$= 3 + 1.80 + 3.196$$

$$= 8\ \text{N/mm}^2$$

**Hence, we understood that $\phi_1 = \left(\frac{53°\,3'}{2}\right)$ defined the major principal plane and therefore $\phi_2 = \left(\frac{126°\,56'}{2}\right)$ should define the minor principal plane.**

**2. For the state stress shown in fig. Find the principal plane and principal stress and maximum shear stress. (AUC Nov/Dec 2011)**



***Solution:***

***Given Data:***

$$\sigma_x = 20 \text{ N/mm}^2$$

$$\sigma_y = 4 \text{ N/mm}^2$$

$$\tau_{xy} = 10 \text{ N/mm}^2$$

$$\text{Major principal stress, } \sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \left(\frac{20 + 4}{2}\right) + \sqrt{\left(\frac{20 - 4}{2}\right)^2 + 10^2}$$

$$= 12 + \sqrt{8^2 + 10^2}$$

$$= 12 + 12.80 = 24.8 \text{ N/mm}^2$$

$$\text{Minor principal stress, } \sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \left(\frac{20 + 4}{2}\right) - \sqrt{\left(\frac{20 - 4}{2}\right)^2 + 10^2}$$

$$= 12 - 12.8 = 0.8 \text{ N/mm}^2$$

Principal stresses are, $24.8 \text{ N/mm}^2$ and $0.8 \text{ N/mm}^2$.

$$\tan 2\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 10}{20 - 4}$$

$$= \frac{20}{16} = 1.25$$

$$2\theta = 51^\circ\, 20'$$

Since $\theta$ is general angle, the specific angles representing the principal planes are designated as $\phi_1$ and $\phi_2$.

$$2\,\phi_1 = 51^\circ\, 20' \text{ and } 128^\circ\, 39'$$

$$\text{Using } 2\,\phi_1 = 51^\circ\, 20'$$

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\phi + \tau \sin 2\phi$$

$$= \left(\frac{20+4}{2}\right) + \left(\frac{20-4}{2}\right)\cos 51° 20' + 10 \sin 51° 20'$$

$$= 12 + 4.99 + 7.8$$

$$= 24.8 \text{ N/mm}^2$$

Hence, we understood that $\phi_1 = \left(\frac{51° 20'}{2}\right)$ defines the major principal plane and $\phi_2 = \left(\frac{128° 39'}{2}\right)$ defines the minor principal plane.

***Maximum Shear Stress***

$$\text{Maximum shear stress} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

$$= \frac{24.8 - 0.8}{2} = 12 \text{ N/mm}^2$$

**3. The rectangular stress components of a point in three dimensional stress system are defined as a $\sigma_x$=20Mpa , $\sigma_y$=- 40Mpa , $\sigma_z$ = 80Mpa , $\tau_{xy}$ =40Mpa , , $\tau_{yz}$= -60Mpa , , $\tau_{xz}$=20Mpa . Determine the principal stresses and principal planes.**

**(AUC Apr/ May 2011)**

***Solution:*** Now, $I_1 = \sigma_x + \sigma_y + \sigma_z$

$$= 20 - 40 + 80 = 60$$

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$= (20)(-40) + (-40)(80) + (80)(20) - 40^2 - (-60)^2 - (20)^2$$

$$= -800 - 3200 + 1600 - 1600 - 3600 - 400$$

$$= -8000$$

$$I_3 = -(\sigma_x \tau_{yz}^2 + \sigma_y \tau_{zx}^2 + \sigma_z \tau_{xy}^2 - \sigma_x \sigma_y \sigma_z - 2\,\tau_{xy}\,\tau_{yz}\,\tau_{zx})$$

$$= -[\,(20)(-60)^2 + (-40)(20)^2 + (80)(40)^2 -$$

$$(20\,(-40)(80) - 2\,(40)(-60)(20)\,]$$

$$= -[\,72000 - 16000 + 128000 + 64000 + 96000\,]$$

$$I_3 = -3{,}44{,}000$$

$\therefore$ Cubic equation becomes,

$$\sigma^3 - 60\,\sigma^2 - 8000\,\sigma + 3{,}44{,}000 = 0$$

$$\text{Now,} \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \qquad \ldots (a)$$

$$\text{Put} \quad \sigma = r\cos\theta + \frac{I_1}{3} = r\cos\theta + 20$$

$\therefore$ The cubic equation becomes,

$$(r\cos\theta + 20)^3 - 60(r\cos\theta + 20)^2 - 8000(r\cos\theta + 20) + 3{,}44{,}000 = 0$$

$$r^3\cos^3\theta + 60r^2\cos^2\theta + 1200r\cos\theta + 8000 - 60r^2\cos^2\theta - 24000 -$$

$$2400r\cos\theta - 8000r\cos\theta - 1{,}60{,}000 + 3{,}44{,}000 = 0$$

$$r^3\cos^3\theta - 9200r\cos\theta - 1{,}76{,}000 = 0$$

$$\cos^3\theta - \frac{9200}{r^2}\cos\theta - \frac{176000}{r^3} = 0 \qquad \ldots (b)$$

Hence, equation $(a)$ and $(b)$ are identical if

$$\frac{9200}{r^2} = \frac{3}{4}$$

$$i.e., \quad r^2 = 12266.667$$

$$r = 110.75$$

$$\frac{176000}{r^3} = \frac{1}{4}\cos 3\theta$$

$$\cos 3\theta = 4\left(\frac{176000}{110.75^3}\right) = 4\left(\frac{176000}{1358411.047}\right)$$

$$\boxed{\cos 3\theta = 0.518}$$

$$\theta_1 = 19.6°, \quad \theta_2 = 100.4°, \quad \theta_3 = 139.6°$$

$$\Rightarrow \theta_2 = 120° - \theta_1$$

$$\theta_3 = 120° + \theta_1$$

$$r_1\cos\theta_1 = 110.75\cos 19.6° = 104.33$$

$$r_2\cos\theta_2 = 110.75\cos 100.4° = -19.99$$

$$r_3\cos\theta_3 = 110.75\cos 139.6° = -84.34$$

$$\sigma_1 = r_1\cos\theta_1 + 20 = 104.33 + 20 = 124.33 \text{ MPa}$$

$$\sigma_2 = r_2\cos\theta_2 + 20 = -19.99 + 20 = 0$$

$$\sigma_3 = r_3\cos\theta_3 + 20 = -84.34 + 20 = -64.34 \text{ MPa}$$

Principal stresses are, 124.33 MPa, 0, and – 64.34 MPa with the principal planes with 19.6°, 100.4° and 139.6°.

4. **A shaft is subjected to a maximum torque of 10KN-m and a maximum bending moment of 8KN-m at perpendicular section. if the allowable equivalent stress in simple is 160MN/m$^2$ , find the diameter of the shaft accoding to the maximum shear stress theory**
**(AUC Nov/ Dec 2011)**

***Solution:***

***Given Data:***

Maximum torque, T = 10 kN-m

Maximum bending moment, M = 8 kN-m

Allowable equivalent stress in simple tension, $\sigma_e$ = 160 MN/m$^2$

Diameter of shaft = ?

Let '$d$' be the diameter of the shaft, then the principal stresses are given by

$$\sigma_1 = \frac{16}{\pi d^3}[ M + \sqrt{M^2 + T^2} ]$$

$$\text{and} \quad \sigma_2 = \frac{16}{\pi d^3}[ M - \sqrt{M^2 + T^2} ]$$

$$\textit{i.e.,} \quad \sigma_1 = \frac{16}{\pi d^3} [8 \times 10^6 + \sqrt{(8 \times 10^6)^2 + (10 \times 10^6)^2} ]$$

$$= \frac{16 \times 10^6}{\pi d^3} [ 8 + \sqrt{64 + 100} ]$$

$$= \frac{16 \times 10^6}{\pi d^3} [ 8 + \overset{20.5}{12.8} ] = \frac{106.01 \times 10^6}{d^3}$$

$$\sigma_2 = \frac{-24.46 \times 10^6}{d^3}$$

$$q_{max} = \sigma_1 - \sigma_2$$

$$= \frac{106.01 \times 10^6}{d^3} - \left[\frac{-24.46 \times 10^6}{d^3}\right] = \frac{130.47 \times 10^6}{2\, d^3}$$

According to the maximum shear stress theory,

$$q_{max} = \frac{\text{Shear stress at the elastic limit}}{\text{Factor of safety}} = \sigma_e$$

$$\frac{130.47 \times 10^6}{2\, d^3} = \frac{160}{2}$$

$$\boxed{d = 93.42 \text{ mm}}$$