QUESTION WITH ANSWERS

DEPARTMENT : CIVIL SEMESTER:IV
SUB.CODE/ NAME: CE 8402 / Strength of Materials - II

UNIT– 1 ENERGY PRINCIPLES
PART - A (2 marks)

1. **Define strain energy.**
   Whenever a body is strained, the energy is absorbed in the body. The energy which is absorbed in the body due to straining effect is known as strain energy. The strain energy stored in the body is equal to the work done by the applied load in stretching the body
2. **Define Resilience. (AUC May/June 2012)**
   The resilience is defined as the capacity of a strained body for doing work on the removal of the straining force. The total strain energy stored in a body is commonly known as resilience.
3. **Define Proof Resilience.**
   The proof resilience is defined as the quantity of strain energy stored in a body when strained up to elastic limit. The maximum strain energy stored in a body is known as proof resilience.
4. **Define Modulus of Resilience. (AUC Nov/Dec 2012)(AUC May/June2012)**
   It is defined as the proof resilience of a material per unit volume.

   $$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}$$
5. **State the two methods for analyzing the statically indeterminate structures.**
   a. Displacement method (equilibrium method (or) stiffness coefficient method
   b. Force method (compatibility method (or) flexibility coefficient method)
6. **Define Castigliano's first theorem. (AUC Nov/Dec 2012) (AUC May/June 2012)**
   It states that the deflection caused by any external force is equal to the partial derivative of the strain energy with respect to that force.
7. **State Castigliano's second Theorem. (AUC May/June 2012)**
   It states that "If U is the total strain energy stored up in a frame work in equilibrium under an external force; its magnitude is always a minimum.
8. **State the Principle of Virtual work. (AUC Apr/May 2011)**
   It states that the workdone on a structure by external loads is equal to the internal energy stored in a structure ($U_e = U_i$)
   Work of external loads = work of internal loads
9. **What down the expression for the strain energy stored in a rod of length l and cross sectional area A subjected in to tensile load ? (AUC Nov/Dec 2010)**
   Strain energy stored

   $$U = W^2 L \;/ 2AE$$

10. State the various methods for computing the joint deflection of a perfect frame.

1. The Unit Load method
2. Deflection by Castiglione's First Theorem
3. Graphical method: Willot – Mohr Diagram

11. State the deflection of the joint due to linear deformation.

$$\delta_v = \sum_{1}^{n} U \times \Delta$$

$$\delta_H = \sum_{1}^{n} U' \times \Delta$$

$$\Delta = \frac{PL}{Ae}$$

U= vertical deflection

U'= horizontal deflection

12. State the deflection of joint due to temperature variation.

$$\delta = \sum_{1}^{n} U \, X \, A$$

$$= U_1\Delta_1 + U_2\,\Delta_2 + \ldots\ldots\ldots\ldots + U_n\,\Delta_n$$

If the change in length (Δ) of certain member is zero, the product U.Δ for those members will be substituted as zero in the above equation.

13. State the deflection of a joint due to lack of fit.

$$\delta = \sum_{1}^{n} U\,\Delta$$

$$= U_1\Delta_1 + U_2\,\Delta_2 + \ldots\ldots\ldots\ldots + U_n\,\Delta_n$$

If there is only one member having lack of fit Δ_1, the deflection of a particular joint will be equal to $U_1\Delta_1$.

14. What is the effect of change in temperature in a particular member of a redundant frame?

When any member of the redundant frame is subjected to a change in temperature, it will cause a change in length of that particular member, which in turn will cause lack of fit stresses in all other members of the redundant frame.

15. State the difference between unit load and strain energy method in the determination of structures.

In strain energy method, an imaginary load P is applied at the point where the deflection is desired to be determined. P is equated to zero in the final step and the deflection is obtained.

In the Unit Load method, a unit load (instead of P) is applied at the point where the deflection is desired.

16. State the assumptions made in the Unit Load method.

1. The external and internal forces are in equilibrium
2. Supports are rigid and no movement is possible
3. The material is strained well within the elastic limit.

17. State the comparison of Castiglione's first theorem and unit load method.

The deflection by the unit load method is given by

$$\delta = \sum_{1}^{n} \frac{PUL}{AE}$$

$$\delta = \sum_{1}^{n} \frac{PL}{AE} \times U$$

$$= \sum_{1}^{n} \Delta \times U \quad \text{----- (i)}$$

The deflection by castigliano's theorem is given by

$$\delta = \sum_{1}^{n} \frac{PL}{AE}\frac{\partial P}{\partial W} \quad \text{--------- (ii)}$$

By comparing (i) & (ii)

$$\frac{\partial P}{\partial W} = U$$

18. State Maxwell's Reciprocal Theorem. (AUC Apr/May 2011) (AUC Apr/May 2010) (AUC Nov/Dec 2010)(AUC Nov/Dec 2013)

The Maxwell's Reciprocal theorem states as " The work done by the first system of loads due to displacements caused by a second system of loads equals the work done by the second system of loads due to displacements caused by the first system of loads.

19. Define degree of redundancy.

A frame is said to be statically indeterminate when the no of unknown reactions or stress components exceed the total number of condition equations of equilibrium.

20. Define Perfect Frame.

If the number of unknowns is equal to the number of conditions equations available, the frame is said to be a perfect frame.

21. State the two types of strain energies.

a. strain energy of distortion (shear strain energy)
b. strain energy of uniform compression (or) tension (volumetric strain energy)

22. State in which cases, Castiglione's theorem can be used.

1. To determine the displacements of complicated structures.
2. To find the deflection of beams due to shearing (or) bending forces (or) bending moments are unknown.
3. To find the deflections of curved beams springs etc.

23. Define Proof stress.

The stress induced in an elastic body when it possesses maximum strain energy is termed as its proof stress.

24. Write the formula to calculate the strain energy due to bending.

$U = \int (M^2 / 2EI)\, dx$ limit 0 to L

Where,
M = Bending moment due to applied loads.
E = Young's modulus
I = Moment of inertia

25. Write the formula to calculate the strain energy due to torsion

$U = \int (T^2/2GJ)\, dx$ limit 0 to L

Where,

T = Applied Torsion
G = Shear modulus or Modulus of rigidity
J = Polar moment of inertia

26. Write the formula to calculate the strain energy due to pure shear

$U = K \int (V^2\ 2GA)dx$ limit 0 to L

Where,
V= Shear load
G = Shear modulus or Modulus of rigidity
A = Area of cross section.
K = Constant depends upon shape of cross section

27. Write down the formula to calculate the strain energy due to pure shear, if shear stressgiven.

$U = \tau^2 V / 2G$
Where,
τ^2 = Shear Stress
G = Shear modulus or Modulus of rigidity
V = Volume of the material.

28. Write down the formula to calculate the strain energy , if the moment value is given

$U = (M^2 L / 2EI)$

Where, M = Bending moment
L = Length of the beam
E = Young's modulus
I = Moment of inertia

29. Write down the formula to calculate the strain energy , if the torsion moment value is given.

$U = T^2 L/2GJ$

Where, T = Applied Torsion
L = Length of the beam
G = Shear modulus or Modulus of rigidity
J = Polar moment of inertia

30. Write down the formula to calculate the strain energy, if the applied tension load is given.

$U = P^2L / 2AE$

31. Find the strain energy per unit volume, the shear stress for a material is given as 50 N/mm². Take G= 80000 N/mm ².

U= τ 2 per unit volume
2G
= 50 ² / (2 x 80000)
= 0.015625 N/mm ². per unit volume.

32. Find the strain energy per unit volume, the tensile stress for a material is given as 15N/mm².

Take = 2 x10 N/mm ².

U = f ² / 2E per unit volume
= (150) ² / (2 x (2x10 ²)
= 0.05625 N/mm ². per unit volume.

Part B (13 marks)

1. Derive the expression for strain energy in Linear Elastic Systems for the following cases. (i) Axial loading (ii) Flexural Loading (moment (or) couple)

(i)<u>Axial Loading</u>

Let us consider a straight bar of Length L, having uniform cross- sectional area A. If an axial load P is applied gradually, and if the bar undergoes a deformation Δ, the work done, stored as strain energy (U) in the body, will be equal to average force (1/2 P) multiplied by the deformation Δ.

Thus $U = ½ P. Δ$ But $Δ = PL / AE$

$U = ½ P. PL/AE = P^2 L / 2AE$ ---------- (i)

If, however the bar has variable area of cross section, consider a small of length dx and area of cross section Ax. The strain energy dU stored in this small element of length dx will be, from equation (i)

$$dU = \frac{P^2\, dx}{2A_x E}$$

The total strain energy U can be obtained by integrating the above expression over the length of the bar.

$$U = \int_0^L \frac{P^2 dx}{2A_x E}$$

(ii) Flexural Loading (Moment or couple)

Let us now consider a member of length L subjected to uniform bending moment M. Consider an element of length dx and let d_i be the change in the slope of the element due to applied moment M. If M is applied gradually, the strain energy stored in the small element will be

$$dU = ½ Md_i$$

But

$$\frac{d_i}{d_x} = \frac{d}{d_x}(dy/dx) = d^2y/d^2x = M/EI$$

$$d_i = \frac{M}{EI} dx$$

Hence $dU = ½ M (M/EI) dx$

$= (M^2/2EI) dx$

Intgrating $U = \int_0^L \frac{M^2 dx}{2EI}$

2. **State and prove the expression for castigliano's first theorem.**

Castigliano's first theorem:

It states that the deflection caused by any external force is equal to the partial derivative of the strain energy with respect to that force. A generalized statement of the theorem is as follows:

" If there is any elastic system in equilibrium under the action of a set of a forces W_1, W_2, W_3 …………W_n and corresponding displacements δ_1, δ_2, δ_3………… δ_n and a set of moments M_1, M_2, M_3………M_n and corresponding rotations Φ_1, Φ_2, Φ_3,…….. Φ_n, then the partial derivative of the total strain energy U with respect to any one of the forces or moments taken individually would yield its corresponding displacements in its direction of actions.

Expressed mathematically,

$$\frac{\partial U}{\partial W_1} = \delta_1 \qquad \text{-------------} \quad (i)$$

$$\frac{\partial U}{\partial M_1} = \phi_1 \qquad \text{-------------} \quad (ii)$$

Proof:

Consider an elastic body as show in fig subjected to loads W_1, W_2, W_3 ………etc. each applied independently. Let the body be supported at A, B etc. The reactions R_A, R_B etc do not work while the body deforms because the hinge reaction is fixed and cannot move (and therefore the work done is zero) and the roller reaction is perpendicular to the displacements of the roller. Assuming that the material follows the Hooke's law, the displacements of the points of loading will be linear functions of the loads and the principles of superposition will hold.

Let δ_1, δ2, δ3……… etc be the deflections of points 1, 2, 3, etc in the direction of the loads at these points. The total strain energy U is then given by

$$U = \tfrac{1}{2}(W_1\delta_1 + W_2\,\delta_2 + \ldots\ldots) \qquad \text{---------} \qquad \text{(iii)}$$

Let the load W_1 be increased by an amount dW_1, after the loads have been applied. Due to this, there will be small changes in the deformation of the body, and the strain energy will be increased slightly by an amount dU. expressing this small increase as the rate of change of U with respect to W_1 times dW_1, the new strain energy will be

$$U + \frac{\partial U}{\partial W_1} x dW_1 \qquad \text{---------} \qquad \text{(iv)}$$

On the assumption that the principle of superposition applies, the final strain energy does not depend upon the order in which the forces are applied. Hence assuming that dW_1 is acting on the body, prior to the application of W1, W_2, W_3………etc, the deflections will be infinitely small and the corresponding strain energy of the second order can be neglected. Now when W1, W_2, W_3………etc, are applied (with dW_1 still acting initially), the points 1, 2, 3 etc will move through δ_1, δ2, δ3……… etc. in the direction of these forces and the strain energy will be given as above. Due to the application of W_1, rides through a distance δ_1 and produces the external work increment $dU = dW_1 . \delta_1$. Hence the strain energy, when the loads are applied is

$$U + dW_1.\delta_1 \qquad \text{-----------} \text{ (v)}$$

Since the final strain energy is by equating (iv) & (v).

$$U + dW_1.\delta_1 = U + \frac{\partial U}{\partial W_1} x dW_1$$

$$\delta_1 = \frac{\partial U}{\partial W_1}$$

Which proves the proportion? Similarly it can be proved that $\Phi_1 = \dfrac{\partial U}{\partial M_1}$.

Deflection of beams by castigliano's first theorem:

If a member carries an axial force the energies stored is given by

$$U = \int_0^L \frac{P^2 dx}{2A_x E}$$

In the above expression, P is the axial force in the member and is the function of external load W_1, W_2,W_3 etc. To compute the deflection δ_1 in the direction of W_1

$$\delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L \frac{P}{AE}\frac{\partial p}{\partial W_1} dx$$

If the strain energy is due to bending and not due to axial load

$$U = \int_0^L \frac{M^2 dx}{2EI}$$

$$\delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L M \frac{\partial M}{\partial W_1} \frac{dx}{EI}$$

If no load is acting at the point where deflection is desired, fictitious load W is applied at the point in the direction where the deflection is required. Then after differentiating but before integrating the fictitious load is set to zero. This method is sometimes known as the fictitious load method. If the rotation Φ_1 is required in the direction of M_1.

$$\Phi_1 = \frac{\partial U}{\partial M_1} = \int_0^L M \frac{\partial M}{\partial M_1} \frac{dx}{EI}$$

3. State and prove the Castigliano's second Theorem.

Castigliano's second theorem:

It states that the strain energy of a linearly elastic system that is initially unstrained will have less strain energy stored in it when subjected to a total load system than it would have if it were self-strained.

$$\frac{\partial u}{\partial t} = 0$$

For example, if λ is small strain (or) displacement, within the elastic limit in the direction of the redundant force T,

$$\frac{\partial u}{\partial t} = \lambda$$

λ=0 when the redundant supports do not yield (or) when there is no initial lack of fit in the redundant members.

Proof:

Consider a redundant frame as shown in fig.in which Fc is a redundant member of geometrical length L.Let the actual length of the member Fc be (L-λ), λ being the initial lack of fit.F2 C represents thus the actual length (L-λ) of the member. When it is fitted to the truss, the member will have to be pulled such that F2 and F coincide.

According to Hooke's law

$F_2 F_1$ = Deformation = $\dfrac{T(l-\lambda)}{AE} = \dfrac{TL}{AE}(approx)$

Where T is the force (tensile) induced in the member.

Hence $FF_1=FF_2-F_1 F_2$

$$\lambda = \frac{TL}{AE} \quad \text{------------------------------------} \ (\text{ i })$$

Let the member Fc be removed and consider a tensile force T applied at the corners F and C as shown in fig.

FF_1 = relative deflection of F and C

$$= \frac{\partial u1}{\partial T} \quad \text{--} \ (\text{ ii })$$

According to castigliano's first theorem where U^1 is the strain energy of the whole frame except that of the member Fc.

Equating (i) and (ii) we get

$$\frac{\partial u1}{\partial T} = \lambda -- \frac{TL}{AE}$$

(or)

$$\frac{\partial u1}{\partial T} + \frac{TL}{AE} = \lambda \quad \text{-----------------------} \ (\text{ iii })$$

To strain energy stored in the member Fc due to a force T is

$$U_{FC} = ½\ T.\ \frac{TL}{AE} = \frac{T2L}{2AE}$$

$$\frac{\partial U_{FC}}{\partial T} = \frac{TL}{AE}$$

Substitute the value of $\frac{TL}{AE}$ in (iii) we get

$$\frac{\partial u'}{\partial T} + \frac{\partial U_{FC}}{\partial T} = \lambda \quad \text{(or)} \quad \frac{\partial U}{\partial T} = \lambda$$

When U= U^1 + U Fc.If there is no initial lack of fit, λ =0 and hence $\frac{\partial U}{\partial T} = 0$

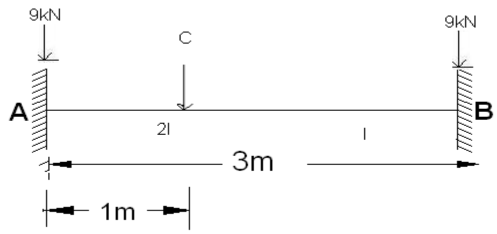
Note:

i) Castigliano's theorem of minimum strain energy is used for the for analysis of statically indeterminate beam ands portal tranes,if the degree of redundancy is not more than two.

ii) If the degree of redundancy is more than two, the slope deflection method or the moment distribution method is more convenient.

4. A beam AB of span 3mis fixed at both the ends and carries a point load of 9 KN at C distant 1m from A. The M.O.I. of the portion AC of the beam is 2I and that of portion CB is I. calculate the fixed end moments and reactions.

Solution:



There are four unknowns M_a, $Ra_,$ M_b and R_b.Only two equations of static are available (ie) $\sum v = 0$ and $\sum M = 0$

This problem is of second degree indeterminacy.

First choose M_A and M_B as redundant.

$$\delta_A = \frac{\partial U_{AB}}{\partial R_A} = 0 = \int \frac{Mx}{EI} \frac{\partial M_x}{\partial R_A} dx \quad \text{-----------(1)}$$

$$\theta_A = \frac{\partial U_{AB}}{\partial M_A} = 0 = \int_A^B \frac{M_x}{EI} \frac{\partial M_x}{\partial M_A} dx \quad \text{-------------(2)}$$

1) For portion AC:

Taking A as the origin

$M_x = -M_A + R_A\, x$

$$\frac{\partial M_x}{\partial R_A} = x; \frac{\partial M_x}{\partial M_A} = -1$$

$M.O.I = 2I$ Limits of x: 0 to 1m

Hence $\int_A^C \frac{M_x}{EI} \frac{\partial M_x}{\partial R_A} dx = \int_0^1 \frac{\left(M_A + R_A\ x\ x\right)}{2EI} dx$

$$= \frac{1}{2EI}\left(\frac{-M_A\left(\;\right)^2}{2} + \frac{R_A\left(\;\right)^3}{3}\right)$$

$$= \frac{1}{2EI}\left(\frac{R_A}{3} - \frac{M_A}{2}\right)$$

And $$\int_A^C \frac{M_x}{EI}\frac{\partial M_x}{\partial R_A}\,dx = \int_0^1 \frac{\left(M_A + R_A\,x\right)\left(-1\right)}{2EI}\,dx$$

$$= \frac{1}{2EI}\left(M_A\left(\;\right) - \frac{R_A\left(\;\right)^2}{2}\right) = \frac{1}{2EI}\left(M_A - \frac{R_A}{2}\right)$$

For portion CB, Taking A as the origin we have

$$M_x = -M_A + R_A X - 9(X-1)$$

$$\frac{\partial M_x}{\partial R_A} = x; \frac{\partial M_x}{\partial M_A} = -1$$

M.O.I = I Limits of x : 1 to 3 m

Hence

$$\int_C^B \frac{M_x}{EI}\frac{\partial M_x}{\partial R_A}\,dx = \int_1^3 \frac{\left(M_A + R_A\,x - 9(x-1)\right)x}{EI}\,dx$$

$$= \frac{1}{EI}\left[-4M_A + \frac{26}{3}R_A - 42\right]$$

And

$$\int_C^B \frac{M_x}{EI}\frac{\partial M_x}{\partial M_A}\,dx = \int_1^3 \frac{\left(M_A + R_A\,x - 9(x-1)\right)\left(-1\right)}{EI}\,dx$$

$$= \frac{1}{EI}\left[2M_A - 4R_A + 18\right]$$

Subs these values in (1) & (2) we get

$$\frac{\partial U_{AB}}{\partial R_A} = 0$$

$$\Rightarrow \frac{1}{EI}\left[\frac{R_A}{3} - \frac{M_A}{2}\right] + \frac{1}{EI}\left[-4M_A + \frac{26}{3}R_A - 42\right] = 0$$

$2.08 - M_A = 9.88$ ____________ (3)

$$\frac{\partial U_{AB}}{\partial M_A} = 0$$

$$\Rightarrow \frac{1}{2EI}\left[\frac{M_A}{1} - \frac{R_A}{2}\right] + \frac{1}{EI}\left[2M_A - 4R_A + 18\right] = 0$$

$M_A - 1.7R_A = -7.2$ -------------- (4)

Solving (3) & (4)

$M_A = 4.8$ KN – M (assumed direction is correct)
$R_A = 7.05$ KN

To find M_B, take moments at B, and apply the condition $\sum M = 0$ there. Taking clockwise moment as positive and anticlockwise moment as negative. Taking M_B clockwise, we have

$M_B - M_A = R_A (3) - 9x2 = 0$

$M_B - 4.8 + (7.05x\ 3) - 18 = 0$
$M_B = 1.65$ KN – m (assumed direction is correct)

To find R_B Apply $\sum V = 0$ for the whole frame.

$R_B = 9 - R_A = 9 - 7.05 = 1.95$ KN

5. Using Castigliano's First Theorem, determine the deflection and rotation of the overhanging end A of the beam loaded as shown in Fig.

Sol:

Rotation of A:

R_B x L = -M
R_B = -M/L

R_B = M/L ($\downarrow$)
& R_C = M/L ($\uparrow$)

$$\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_A^B M_x.\frac{\partial M_x}{\partial M}dx + \frac{1}{EI}\int_C^B M_x.\frac{\partial M_x}{\partial M}.dx \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

For any point distant x from A, between A and B (i.e.) x = 0 to x = L/3

$M_x = M$; and $\dfrac{\partial M_x}{\partial M} = 1$ _________ (2)

For any point distant x from C, between C and B (i.e.) x = 0 to x = L

$M_x = (M/L)\,x$; and $\dfrac{\partial M_x}{\partial M} = \dfrac{x}{L}$ _________ (3

Subs (2) & (3) in (1)

$$\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_0^{L/3} M\ (1).dx + \frac{1}{EI}\int_0^{L}\left(\frac{M}{L}x\right)\frac{x}{L}dx$$

$$= \frac{ML}{3EI} + \frac{ML}{3EI}$$

$$= \frac{2ML}{3EI}(clockwise)$$

b) Deflection of A:

To find the deflection at A, apply a fictitious load W at A, in upward direction as shown in fig.

$$R_B x L = -(M + \frac{4}{3}WL)$$

$$R_B = -(M + \frac{4}{3}WL)\frac{1}{L} \qquad R_B = (M + \frac{4}{3}WL)\frac{1}{L}$$

$$R_C = (M + \frac{1}{3}WL)\frac{1}{L}$$

$$\delta_A = \frac{\partial U}{\partial W} = \frac{1}{EI}\int_A^B M_x\frac{\partial M_x}{\partial W} + \frac{1}{EI}\int_C^B M_x\frac{\partial M_x}{\partial W}.dx$$

For the portion AB, x = 0 at A and x = L/3 at B

$M_x = M + W_x$

$$\frac{\partial M_x}{\partial W} = x$$

For the portion CB, x = 0 at C and x = L at B

$$M_x = \left(M + \frac{1}{8}WL\right)\frac{1}{L}.x$$

$$\frac{\partial M_x}{\partial W} = \frac{x}{3}$$

$$\delta_A = \frac{1}{EI}\int_0^{L/3}(M + Wx)x + \frac{1}{EI}\int_0^{L}\left(M + \frac{1}{3}WL\right)\frac{x}{L}.\frac{x}{3}dx$$

Putting W = 0

$$\delta_A = \frac{1}{EI}\int_0^{L/3}(Mx)dx + \frac{1}{EI}\int_0^{L}\left(\frac{Mx^2}{3L}\right)dx$$

$$\delta_A = \frac{M}{EI}\left(\frac{x^2}{2}\right)_0^{L/3} + \frac{M}{3EI}\left(\frac{x^3}{3}\right)_0^{L}$$

$$\delta_A = \frac{ML^2}{18EI} + \frac{ML^2}{9EI}$$

$$\delta_A = \frac{ML^2}{6EI}$$

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6. Using the principle of least work, analyze the portal frame shown in Fig. Also plot the B.M.D.

Sol:

The support is hinged. Since there are two equations at each supports. They are H_A, V_A, H_D, and V_D. The available equilibrium equation is three. (i.e.) $\sum M = 0, \sum H = 0, \sum V = 0$.

$\therefore$ The structure is statically indeterminate to first degree. Let us treat the horizontal H ($\leftarrow$) at A as redundant. The horizontal reaction at D will evidently be = (3-H) ($\leftarrow$). By taking moments at D, we get

$(V_A \times 3) + H(3-2) + (3 \times 1)(2 - 1.5) - (6 \times 2) = 0$

$V_A = 3.5 - H/3$

$V_D = 6 - V_A = 2.5 + H/3$

By the theorem of minimum strain energy,

$$\frac{\partial U}{\partial H} = 0$$

$$\frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BE}}{\partial H} + \frac{\partial U_{CE}}{\partial H} + \frac{\partial U_{DC}}{\partial H} = 0$$

(1)For member AB:

Taking A as the origin.

$$M = \frac{-1.x^2}{2} + H.x$$

$$\frac{\partial M}{\partial H} = x$$

$$\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI}\int_0^3 M \frac{\partial M}{\partial H} dx$$

$$= \frac{1}{EI}\int_0^3 \left(\frac{-x^2}{2} + Hx\right)x \; dx$$

$$= \frac{1}{EI}\left[\frac{Hx^3}{3} - \frac{x^4}{8}\right]_0^3$$

$$= \frac{1}{EI}\left[H - 10.12\right]$$

(2) For the member BE:

Taking B as the origin.

$$M = (H \; x \; 3) - (3 \; x \; 1 \; 1.5) + \left(3.5 \quad \frac{H}{3}\right)x$$

$$M = 3H - 4.5 + 3.5x - \frac{Hx}{3}$$

$$\frac{\partial M}{\partial H} = 3 - \frac{x}{3}$$

$$\frac{\partial U_{BE}}{\partial H} = \frac{1}{EI}\int_0^1 M \frac{\partial M}{\partial H} dx$$

$$= \frac{1}{EI}\int_0^1 \left(3H - 4.5 + 3.5x - \frac{Hx}{3}\right)\left(3 - \frac{x}{3}\right)dx$$

$$= \frac{1}{EI}\int_0^1 \left(9H - 13.5 + 10.5x - Hx - Hx + 1.5x - 1.67x^2 + \frac{Hx^2}{9}\right)dx$$

$$= \frac{1}{EI}\int_0^1 \left(9H - 13.5 + 12x - 2Hx - 1.67x^2 + \frac{Hx^2}{9}\right)dx$$

$$= \frac{1}{EI}\left(9Hx - 13.5x + 6x^2 - Hx^2 - 0.389x^3 + \frac{Hx^3}{27}\right)_0^1 = \frac{1}{EI}\left(9H - 13.5 + 6^2 - H - 0.389 + \frac{H}{27}\right)$$

$$= \frac{1}{EI}\left[8H - 7.9\right]$$

(3) For the member CE:

Taking C as the origin

$$M = -(3 - H)x2 + (2.5 + \frac{H}{3})x$$

$$M = -6 + 2H + 2.5x + \frac{Hx^3}{3}$$

$$\frac{\partial U_{CE}}{\partial H} = \frac{1}{EI}\int_0^2 M \frac{\partial M}{\partial H}$$

$$= \frac{1}{EI}\int_0^2\left[\left(-6 + 2H + 2.5x + \frac{Hx}{3}\right)\left(2 + \frac{x}{3}\right)\right]$$

$$= \frac{1}{EI}\int_0^2 \left[-12 + 4H + 5x + 6.67Hx - 2x + 6.67Hx + 0.833x^2 + \frac{Hx^2}{9}\right]dx$$

$$= \frac{1}{EI}\int_0^2 \left[-12 + 4H + 3x + 13.34Hx - 2x + 0.833x^2 + \frac{Hx^2}{9}\right]dx$$

$$= \frac{1}{EI}(10.96H - 15.78)$$

(4) For the member DC:

Taking D as the origin

$$M = -(3 - H)x = -3x + Hx$$

$$\frac{\partial M}{\partial x} = x$$

$$\frac{\partial U_{DC}}{\partial H} = \frac{1}{EI}\int_0^2 M \frac{\partial M}{\partial H} dx$$

$$= \frac{1}{EI}\int_0^2 (-3x + Hx)(x)\,dx \qquad = \frac{1}{EI}\int_0^2 (-3x^2 + Hx^2)\,dx$$

$$= \frac{1}{EI}\left(\frac{-3x^3}{3} + \frac{Hx^3}{3}\right)_0^2 dx \qquad = \frac{1}{EI}\left(-x^3 + \frac{Hx^3}{3}\right)_0^2 dx$$

$$= \frac{1}{EI}(2.67H - 8)$$

Subs the values

$$\frac{\partial U}{\partial H} = 0$$

1/EI (9-10.2) + (8.04H-7.9) + (10.96H-15.78) + (-8+2.67H) = 0

30.67H = 41.80

H = 1.36 KN

Hence

V_A = 3.5 - H/3 = 3.5 - 1.36/3 = 3.05 KN

V_D = 2.5 + H/3 = 2.5 + 1.36/3 = 2.95 KN

$M_A = M_D = 0$

$M_B = (-1 \times 3^2)/2 + (1.36 \times 3) = -0.42$ KN –m

$M_C = -(3-H)\,2 = -(3-1.36)2 = -3.28$KNm

7. A simply supported beam of span 6m is subjected to a concentrated load of 45 KN at 2m from the left support. Calculate the deflection under the load point. Take E = 200 x 10^6 KN/m^2 and I = 14 x 10^{-6} m^4.

Solution:

Taking moments about B.

V_A x 6 – 45 x 4=0

V_A x 6 -180 = 0

V_A = 30 KN

V_B = Total Load – V_A = 15 KN

Virtual work equation:

$$|\delta_c|_V = \int_0^L \frac{mMdx}{EI}$$

Apply unit vertical load at c instead of 45 KN

R_A x 6-1 x 4 =0

R_A = 2/3 KN

R_B = Total load –R_A = 1/3 KN

Virtual Moment:

Consider section between AC

$M_1 = 2/3\ X_1$ [limit 0 to 2]

Section between CB

$M_2 = 2/3\ X_2 - 1\ (X_2 - 2)$ [limit 2 to 6]

Real Moment:

The internal moment due to given loading

$M_1 = 30 \times X_1$

$M_2 = 30 \times X_2 - 45\ (X_2 - 2)$

$$(\delta_c)_v = \int_0^2 \frac{m_1 M_1 dx_1}{EI} + \int_2^6 \frac{m_2 M_2 dx_2}{EI}$$

$$= \int_0^2 \frac{\left(\frac{2x_1}{3}\right)(30x_1)}{EI} dx_1 + \int_2^6 \frac{\left(\frac{2}{3}x_2 - (x_2 - 2)\right)(30x_2 - 45(x_2 - 2))}{EI} dx_2$$

$$= \frac{1}{EI}\int_0^2 20x_1^2 + \int_2^6 \left(\frac{2}{3}x_2 - x_2 + 2\right)(30x_2 - 45x_2 + 90)dx_2$$

$$= \frac{1}{EI}\int_0^2 20x_1^2 + \int_2^6 \left(\frac{-x_2}{3} + 2\right)(-15x_2 + 90)dx_2$$

$$= \frac{1}{EI}\int_0^2 20x_1^2 + \int_2^6 5x_2^2 - 30x_2 - 30x_2 + 180dx_2$$

$$= \frac{1}{EI}\left[\frac{20x_1}{3}\right]_0^3 + \left[\frac{5x_2^3}{3} - \frac{60x_2^3}{2} + 180x_2\right]_2^6$$

$$= \frac{20}{EI}\left(\frac{8}{3}\right) + \frac{1}{EI}\left(\frac{5}{3}(6^3 - 2^3) - 30(6^2 - 2^2) + 180(6 - 21)\right)$$

$$= \frac{1}{EI}\left[3.33 + 346.67 - 960 + 720\right]$$

$$= \frac{160}{EI} = \frac{160}{200x10^{6}x14x10^{-6}} = 0.0571\ m\ (or)\ 57.1\ mm$$

The deflection under the load = 57.1 mm

8. Define and prove the Maxwell's reciprocal theorem.

The Maxwell's reciprocal theorem stated as " The work done by the first system loads due to displacements caused by a second system of loads equals the work done by the second system of loads due to displacements caused by the first system of loads"

Maxwell's theorem of reciprocal deflections has the following three versions:

1. The deflection at A due to unit force at B is equal to deflection at B due to unit force at A.

   $\delta_{AB} = \delta_{BA}$
2. The slope at A due to unit couple at B is equal to the slope at B due to unit couple A

   $\Phi_{AB} = \Phi_{BA}$
3. The slope at A due to unit load at B is equal to deflection at B due to unit couple.

   $$\phi_{AB} = \delta'_{AB}$$

Proof:

By unit load method,

$$\delta = \int \frac{Mmdx}{EI}$$

Where,

M= bending moment at any point x due to external load.

m= bending moment at any point x due to unit load applied at the point where deflection is required.

Let m_{XA}= bending moment at any point x due to unit load at A

Let m_{XB} = bending moment at any point x due to unit load at B.

When unit load (external load) is applied at A,

$M = m_{XA}$

To find deflection at B due to unit load at A, apply unit load at B.Then m= m_{XB}

Hence,

$$\delta_{BA} = \int \frac{Mmdx}{EI} = \int \frac{m_{XA}.m_{XB}}{EI}dx \qquad \text{(i)}$$

Similarly,

When unit load (external load) is applied at B, M=m_{XB}

To find the deflection at A due to unit load at B, apply unit load at A.then m= m_{XA}

$$\delta_{AB} = \int \frac{Mmdx}{EI} = \int \frac{mB.m_{XA}}{EI}dx \qquad \text{(ii}$$

Comparing (i) & (ii) we get

$\delta_{AB} = \delta_{BA}$