Unit - 4

Initial Value Problems for ODE

① find $y(0.1)$ if $\frac{dy}{dx} = 1+y$, $y(0)=1$ using Taylor series Method.

Soln $f(x,y) = 1+y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$y' = 1+y$ $\quad y_0' = 1+y_0 = 1+1 = 2$

$y'' = y'$ $\quad y_0'' = y_0' = 2$

$y''' = y''$ $\quad y_0''' = y_0'' = 2$

$$y_1 = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0'''$$

$$= 1 + \frac{(x-0)}{1}(2) + \frac{(x-0)^2}{2}(2) + \frac{(x-0)^3}{6}(2)$$

$$= 1 + 2x + x^2 + \frac{x^3}{3}$$

$$y(0.1) = 1 + 2(0.1) + (0.1)^2 + \frac{(0.1)^3}{3}$$

$$y(0.1) = 1.2103.$$

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② State the fourth order Runge-kutta Algorithm.

$K_1 = hf(x,y)$

$K_2 = hf[x+\frac{h}{2}, y+\frac{K_1}{2}]$

$K_3 = hf[x+\frac{h}{2}, y+\frac{K_2}{2}]$

$K_4 = hf(x+h, y+K_3]$

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③ find $y(1.1)$ if $y' = x+y$, $y(1) = 0$ by Taylor Series Method.

Soln $f(x,y) = x+y$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$

$y' = x+y$ $\quad y_0' = x_0 + y_0 = 1+0 = 1$

$y'' = 1+y'$ $\quad y_0'' = 1 + y_0' = 1+1 = 2$

$y''' = y''$ $\qquad y_0''' = y_0'' = 2$

$y^{IV} = y'''$ $\qquad y_0^{IV} = y_0''' = 2$

$$y_1 = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y^{IV}$$

$$= 0 + \frac{x-1}{1}(1) + \frac{(x-1)^2}{2}(2) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(2)$$

$$y(1.1) = \frac{0.1}{1} + \frac{(0.1)^2}{2} 2 + \frac{(0.1)^3}{6} \times 2 + \frac{(0.1)^4}{24}(2)$$

$y(1.1) = 0.1103.$

(4) State Euler's formula

$$y_{n+1} = y_n + h f(x_n, y_n).$$

(5) State Adam's predictor corrector formula.

$$y_{n+1,p} = y_n + \frac{h}{24}[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

$$y_{n+1,c} = y_n + \frac{h}{24}[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

(6) Using Euler's Method find the solution of the initial Value Problem $y' = y - x^2 + 1$, $y(0) = 0.5$ at $x = 0.2$ taking $h = 0.2$

Soln

$f(x,y) = y - x^2 + 1$, $x_0 = 0$, $y_0 = 0.5$, $h = 0.2$

$x_1 = 0.2$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 0.5 + 0.2 f(0, 0.5)$$

$$= 0.5 + 0.2 [1.5]$$

$$= 0.5 + 0.3$$

$y(0.2) = 0.8$

(7) State the advantages and disadvantages of the Taylor's Series method.

Merits :

* Taylor formula is easily derived for any order according to our interest
* The values of $y(x)$ for any x (x need not be at grid points) are easily obtained.

Demerit :

This method suffers from the time consumed in calculating the higher derivatives.

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(8) State the Milne's Predictor and corrector formula.

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_{n}\right)$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_{n} + y'_{n+1}\right]$$

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(9) State the advantages of Rk Method over Taylor series method?

The Rk Methods are designed to give greater accuracy and they possess the advantages of requiring only the function values at some selected points on the Sub interval.

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(10) Using Euler's Method find $y(0.2)$ from $\frac{dy}{dx} = x+y$, $y(0)=1$, with $h=0.2$

Soln

$f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.2)\,[0+1]$$
$$= 1 + 0.2(1)$$
$$y(0.2) = 1.2$$

(11) Given $y' = x+y$, $y(0)=1$ find $y(0.1)$ by Euler's Method.

A:) Soln

$$f(x,y) = x+y, \quad x_0 = 0, \quad y_0 = 1 \qquad h = 0.1$$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$
$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.1)\, f(0,1)$$
$$= 1 + 0.1\,(1)$$
$$= 1 + 0.1$$
$$y(0.1 = 1.1$$

Unit IV
Initial value Problems for ordinary differential Equations

Part-B

① Solve $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0)=2$ by Euler's Method by choosing $h = 0.2$, find $y(0.2)$ and $y(0.4)$

Soln

Hints:

$x_0 = 0$, $y_0 = 2$, $h = 0.2$

$f(x,y) = \log_{10}(x+y)$

Euler's formula

$y_{n+1} = y_n + h f(x_n, y_n)$, $n = 0,1,2,3,\ldots$

put $n=0$ $\quad y_1 = 2 + 0.2 \log_{10}(0+2) = 2.0602 = y(0.2)$

Put $n=1$ $\quad y_2 = 2.0602 + 0.2 \log_{10}(0.2 + 2.0602)$

$= 2.1310 = y(0.4)$

② Solve $\frac{dy}{dx} = -\frac{y^2}{1+x}$ $y(0)=1$ by Euler's Method by choosing $h=0.1$, find $y(0.1)$ and $y(0.2)$

Hints:

Given $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$f(x, y) = \frac{-y^2}{1+x}$$

Euler's formula is $y_{n+1} = y_n + h f(x_n, y_n)$

$n = 0, 1, 2, 3, \ldots$

Put $n = 0$ we get

$$y_1 = y(0.1) = 1 - 0.1\left(\frac{1^2}{1+0}\right) = 0.9$$

Put $n = 1$, we get

$$y_2 = y(0.2) = 0.9 - 0.1\left(\frac{0.9^2}{1+0.1}\right)$$

$$= 0.82636$$

(3) find the value of y at $x = 0.1$ from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$ by Taylor Series method.

Hints

Given $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$y' = x^2y - 1$$

Taylor series expansion is

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \dots$$

$$n=0, \quad y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$y' = x^2 y - 1 \qquad y_0' = -1$

$y'' = 2xy + x^2 y' \qquad y_0'' = 0$

$y''' = 2y + 4xy' + x^2 y'' \qquad y_0''' = 2$

$y^{iv} = 6y' + 6xy'' + x^2 y''' \qquad y_0^{iv} = -6$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(-6)$$

$$= 0.900305$$

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(4) Solve $\frac{dy}{dx} = \sin x + \cos y$, $y(2.5) = 0$ by Modified Euler's Method by choosing $h = 0.5$, find $y(3.5)$

Hints:

$x_0 = 2.5, \ y_0 = 0, \ h = 0.5$

$f(x, y) = \sin x + \cos y$

$$y_{n+1} = y_n + h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)]$$

Put $n=0$

we get $y_1 = y_0 + 0.5\, [f(x_0 + \frac{0.5}{2},\ y_0 + \frac{0.5}{2} f(x_0, y_0))]$

$y_1 = 0 + 0.5\, [f(2.5 + 0.25 + 0.25\, [f(2.5, 0)]]$

$= 0.6354$

Put $n=1$

$y_2 = y_1 + h\, [f(x_1 + \frac{0.5}{2},\ y_1 + \frac{0.5}{2} (f(x_1, y_1))]$

$y_1 = 0.6354 + 0.5\, [f(3 + 0.25,\ 0.6354 + 0.25\, [f(3, 0.6354)]]$

$= 0.93155$

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(5) Apply R.K Method to find an approximate value of y when $x = 0.2$ given that

$\frac{dy}{dx} = x + y,\ y(0) = 1$

Hints

Gn $x_0 = 0,\ y_0 = 1,\ h = 0.2$

$f(x,y) = x + y.$

$x_2 = 0.4,\ y_2 = 0.0795 \quad f_2 = 0.3937$

$x_3 = 0.6,\ y_3 = 0.1762 \quad f_3 = 0.56895$

$y_4^{(p)} = y(0.8) = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 0 + \frac{4(0.2)}{3}[(2\times 0.1996) - 0.39372 + 2\times 0.56895)$

$= 0.30491$

$f_4 = f(0.8, 0.30491) = 0.7070$

$y_4^{(c)} = y(0.8) = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4)$

$= 0.30461$

finding $y(1.0)$

$x_1 = 0.2 \quad y_1 = 0.02 \quad f_1 = 0.1996$

$x_2 = 0.4 \quad y_2 = 0.0795 \quad f_2 = 0.3937$

$x_3 = 0.6 \quad y_3 = 0.1762 \quad f_3 = 0.56895$

$x_4 = 0.8 \quad y_4 = 0.3046 \quad f_4 = 0.7072$

$y_4^{(p)} = y(1.0) = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$

$= 0.02 + \frac{4(0.2)}{3}[(2\times 0.3937) - 0.56895 + 2\times 0.7072)$

$= 0.45544$

$f_5 = f(1.0, 0.45544)$

$= 0.7926$

$y_5^{(c)} = y(1.0) = y_3 + \frac{h}{3}[f_3 + 4f_4 + f_5]$

$= 0.56895 + \frac{0.2}{3}[0.56895 + 4\times 0.7072 + 0.7926]$

$= 0.4556$

(6) Using Milnes Method to find $y(4.4)$ gn that given that $5xy' - y^2 - 2 = 0$ $y(4) = 1$, $y(4.1) = 1.0049$ $y(4.2) = 1.0097$ $y(4.3) = 1.0143$

Hints:

$y' = \frac{2-y^2}{5x}$, $x_0 = 4$, $x_1 = 4.1$, $x_2 = 4.2$

$x_3 = 4.3$ $x_4 = 4.4$

$y_0 = 1$, $y_1 = 1.0049$

$y_2 = 1.0097$ $y_3 = 1.0143$

$x_2 = 0.4, \quad y_2 = 0.0795 \quad f_2 = 0.3937$

$x_3 = 0.6, \quad y_3 = 0.1762 \quad f_3 = 0.56895$

$$y_4^{(p)} = y(0.8) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[(2 \times 0.1996) - 0.39372 + 2 \times 0.56895\right)$$

$$= 0.30491$$

$$f_4 = f(0.8, 0.30491) = 0.7070$$

$$y_4^{(c)} = y(0.8) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right)$$

$$= 0.30461$$

finding $y(1.0)$

$x_1 = 0.2 \quad y_1 = 0.02 \quad f_1 = 0.1996$

$x_2 = 0.4 \quad y_2 = 0.0795 \quad f_2 = 0.3937$

$x_3 = 0.6 \quad y_3 = 0.1762 \quad f_3 = 0.56895$

$x_4 = 0.8 \quad y_4 = 0.3046 \quad f_4 = 0.7072$

$$y_4^{(p)} = y(1.0) = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 0.02 + \frac{4(0.2)}{3}\left[(2 \times 0.3937) - 0.56895 + 2 \times 0.7072\right)$$

Sub these values

$y_1 = 0.0493$, $y_2 = 0.0467$ $y_3 = 0.0452$

By Milne's Method,

$$y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 1.01897$$

$$y_4' = 0.0437$$

By Milne's ~~predictor~~ corrector formula

$$y_{4,C} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.01874$$

$$y_4 = y(4.4) = 1.01874$$

(7) Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$, $y(1.3) = 1.979$

$y(1.1) = 1.233$ $y(1.2) = 1.548$

Evaluate $y(1.4)$ by Adams bashforth

Hints:

Given $x_0 = 1$, $y_0 = 1$, $x_1 = 1.1$, $y_1 = 1.233$

$x_2 = 1.2$, $y_2 = 1.548$, $x_3 = 1.3$

$y_3 = 1.979$

$y_0' = 2, \quad y_1' = 2.70193 \quad y_2' = 3.66912$

$y_3' = 5.03451$

To find $y_4 = y(x_4) = y(1.4)$

Predictor Method:

$$y_{4,P} = y(1.4) = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[(55 \times 5.03451) - (59 \times 3.66912) + (37 \times 2.70193) - (9 \times 2)\right]$$

$$= 2.57229$$

We compute $y_{4,P}' = 7.0017$

Corrector Method:

$$y_{4,c} = y(1.4) = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.979 + \frac{0.1}{24}\left[(9 \times 7.0017) + (19 \times 5.03451) - (5 \times 3.66912) + 2.70193\right]$$

$$= 2.57495$$

(8) find the value of $y(0.4)$ by Milne's Method for $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ Use Taylor's Series to get the value of y at $x = 0.1$, Euler's Method for y at $x = 0.2$ and Runge-kutta 4th order Method for y at $x = 0.3$.

Hints

find $y_1 = y(0.1)$ by Taylor's Method

$x_0 = 0$, $y_0 = 1$, $h = 0.1$ $\Rightarrow y' = xy + y^2$

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \dots$$

$n = 0$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$y' = xy + y^2$ $\quad y_0' = 1$

$y'' = xy' + y + 2yy'$ $\quad y_0'' = 3$

$y''' = xy'' + 2y' + 2yy'' + 2y'^2$ $\quad y_0''' = 10$

$y_1 = y(0.1) = 1.1167$

To find $y_2 = y(0.2)$ by Euler's Method

$x_1 = 0.1$, $y_1 = 1.1167$

Gn $f(x,y) = y' = xy + y^2$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$n = 1$

$y_2 = y_1 + h\, f(x_1, y_1)$

$y_2 = y(0.2) = 1.1167 + (0.1)(0.1)(1.1167) + (1.1167)^2$

$= 1.2526$

To find $y_3 = y(0.3)$

$x_2 = 0.2$, $y_2 = 1.2526$, $h = 0.1$

$f(x,y) = xy + y^2$

$y_3 = y(0.3) = y_2 + \Delta y$

$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

By Milnes corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.7944$$

$$y_4 = y(0.4) = 1.7944$$

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(9) Consider the initial value problem $\frac{dy}{dx} = y - x^2$
$y(0) = 1$
i) find $y(0.1)$ and $y(0.2)$ by R.K Method of order 4
ii) find $y(0.3)$ by Euler's Method

Hints

$$\frac{dy}{dx} = y - x^2$$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$ $f(x,y) = y - x^2$

To find $y(0.1) = y_1$ by R-K Method

$$y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0) = (0.1)f(0,1) = 0.1$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.10475$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1049875$$

$k_1 = hf(x_2, y_2) = 0.1819$

$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.2141$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.21886$

$k_4 = hf(x_2 + h, y_2 + k_3) = 0.2607$

$\Delta y = 0.2181$

$y_3 = 1.2526 + 0.2181 = 1.4707 = y(0.3)$

To find $y_4 = y(0.4)$ by Milnes Method.

$y' = xy + y^2 \quad x_0 = 4, \; x_1 = 0.1, \; x_2 = 0.2$

$x_3 = 0.3, \; x_4 = 0.4, \; y_0 = 1$

$y_1 = 1.1167, \; y_2 = 1.2526, \; y_3 = 1.4707$

Sub these values

$y_0' = 1, \; y_1' = 1.3587, \; y_2' = 1.8195 \; y_3' = 2.6042$

By Milnes Predictor formula

$y_4 = 1.8142$

$y_4' = x_4 y_4 + y_4^2 = 4.017$

$k_4 = hf(x_0+h,\ y_0+k_3) = 0.10949875$

$y_1 = y(0.2) = 1 + \frac{1}{6}(0.1 + 2(0.10475)$
$+ 2(0.1049875) + 0.10949875)$

$= 1.10483$

To find $y_2 = y(0.2)$ by RK Method of order 4.

$\frac{dy}{dx} = y - x^2 = f(x,y)$

$x_1 = 0.1 \quad y_1 = 1.10483 \quad h = 0.1$

$k_1 = hf(x_1, y_1) = (0.1)(y_1 - x^2)$

$= (0.1)[1.10483 - 0.1^2]$

$= 0.109483$

$k_2 = hf(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}]$

$= 0.11371$

$k_3 = hf(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}) = 0.11392$

$k_4 = hf(x_1 + h,\ y_1 + k_3] = 0.117875$

$$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 0.11377$$

$$y_2 = y_1 + \Delta y$$

$$y_2 = 1.2186$$

ii) To find $y_3 = y(0.3)$ by Euler's Method

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$n=2$ $$y_3 = y_2 + h f(x_2, y_2)$$

$$y_3 = y(0.3)$$

$$= 01.2186 + 0.1(1.2186 - 0.2^2)$$

$$= 1.33646.$$