

Unit-2 Interpolation and Approximation

Part -A

- 1) State Newton's backward (interpolation) formula [Nov/12] [Nov/14]

$$y = y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \dots$$

Where $v = \frac{x - x_n}{h}$

- 2) State Newton's forward interpolation formula [May/13] [Nov/13]

$$y = f(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots$$

Where $u = \frac{x - x_0}{h}$

- 3) Using Lagrange's formula find the polynomial to the given data

x	0	1	3
y	5	6	50

[May/13]

Sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)} (5) + \frac{(x-0)(x-3)}{(1-0)(1-3)} (6) + \frac{(x-0)(x-1)}{(3-0)(3-1)} (50)$$

$$= \frac{5}{3} (x^2 - 4x + 3) - 3 (x^2 - 3x) + \frac{50}{6} (x^2 - x)$$

$$y = f(x) = x^2 \left(\frac{5}{3} - 3 + \frac{50}{6} \right) + x \left(-\frac{20}{3} + 9 - \frac{50}{6} \right) + \frac{15}{3}$$

$$= 7x^2 - 6x + 5.$$

- 4) Find the divided differences table for $f(x) = x^3 - x^2 + 3x + 8$ for the arguments 0, 1, 4, 5.

Sol:

Given $f(x) = y = x^3 - x^2 + 3x + 8$

x	0	1	4	5
y	8	11	68	123

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	8	$\frac{11-8}{1-0} = 3$	$\frac{19-3}{4-0} = 4$	$\frac{9-4}{5-0} = 1$
1	11	$\frac{68-11}{4-1} = 19$	$\frac{55-19}{5-1} = 9$	
4	68	$\frac{123-68}{5-4} = 55$		
5	123			

- 5) Find the second degree polynomial through the points $(0,2), (2,1), (1,0)$ using Lagrange's formula [Nov/14]

Sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Here $x_0 = 0$; $x_1 = 2$; $x_2 = 1$; $y_0 = 2$; $y_1 = 1$; $y_2 = 0$

$$y = \frac{(x-2)(x-1)}{(0-1)(0-1)} (2) + \frac{(x-0)(x-1)}{(2-0)(2-1)} (1) + \frac{(x-0)(x-2)}{(1-0)(1-2)} (0)$$

$$= 2(x^2 - 3x + 2) + 2(x^2 - x) + 0$$

$$= 2x^2(2+2) + x(-6-2) + 4$$

$$y = 4x^2 - 8x + 4$$

- 6) For cubic splines, what are the 4n conditions required to evaluate the unknowns [Nov/15]

1. The function values must be equal at the interior knots $(2n-2)$ condition.
2. The first & last functions must pass through the end points (2 condition)
3. The first derivatives at the interior knots must be equal $(n-1)$ condition
4. The second derivatives at the interior knots must be equal $(n-1)$ condition.
5. The second derivatives at the end knots are zero (2 condition)

(iii) $x(1,4)$ $(3,10)$ & $(4,185)$ [Nov 15]

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	$\frac{4-1}{1-0} = 3$	$\frac{18-3}{3-0} = 5$	$\frac{9-5}{4-0} = 1$
1	4	$\frac{40-4}{3-1} = 18$		
3	40	$\frac{85-40}{4-3} = 45$		
4	85			

f) Show that $\Delta_{bcd}^3 \left(\frac{1}{a} \right) = \frac{-1}{abcd}$

Sol:

Let $f(x) = \frac{1}{x}$

x	a	b	c	d
$f(x)$	$\frac{1}{a}$	$\frac{1}{b}$	$\frac{1}{c}$	$\frac{1}{d}$

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
a	$\frac{1}{a}$	$\frac{\frac{1}{b} - \frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)} = \frac{-1}{ab}$	$\frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a} = \frac{1}{abc}$	
b	$\frac{1}{b}$	$\frac{\frac{1}{c} - \frac{1}{b}}{c-b} = \frac{b-c}{bc(c-b)} = \frac{-1}{bc}$		$\frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{-1}{abcd}$
c	$\frac{1}{c}$	$\frac{\frac{1}{d} - \frac{1}{c}}{d-c} = \frac{c-d}{dc(d-c)} = \frac{-1}{cd}$	$\frac{-\frac{1}{cd} + \frac{1}{bc}}{d-b} = \frac{1}{bcd}$	
d	$\frac{1}{d}$			

1) Define cubic spline which is commonly used for interpolation? [May 14]

- i) $S(x)$ is a polynomial of degree 1 in $x < x_0$ and $x > x_n$
- ii) $S(x)$ is at most a cubic polynomial in $(x_{i-1} - x_i)$
- iii) $S(x)$, $S'(x)$ & $S''(x)$ are continuous at each point
- iv) $S(x_i) = y_i \quad i = 0, 1, 2, 3, \dots, n.$

Unit II
Interpolation and Approximation.

- ① Find the missing term in the following table using Lagrange's interpolation.

X	0	1	2	3	4
Y	1	3	9	-	81

Hints:

$$\begin{aligned}
 y = f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 \\
 &+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \\
 &= \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} (1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} (3) \\
 &+ \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} (9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} (81)
 \end{aligned}$$

$$f(3) = 31$$

② Using Lagrange's formula find the Polynomial for the following data

x	0	1	2	4
$f(x)$	2	3	12	147

Hint:

$$\begin{aligned}
 y = f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 \\
 &+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 \\
 &+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3
 \end{aligned}$$

$$f(x) = \frac{1}{8} [31x^3 - 61x^2 - 38x + 16]$$

③ Using Lagrange's inverse interpolation formula, find the value of x when $y=0$ from the given data
 $f(30) = -30, f(34) = -13, f(38) = 3$

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \cdot x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \cdot x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \cdot x_2$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \cdot x_3$$

$$x = -0.78208 + 6.53225 + 33.681818 - 2.20162$$

$$x = 37.2304$$

④ Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula and hence find $f(8)$

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48	52			
5	100		15		
7	294	97		1	0
10	900	202		1	
11	1210	310			0
13	2028	409		1	

$$\begin{aligned}
 f(x) &= f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) \\
 &\quad + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) \\
 &\quad + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4) \\
 f(x) &= 48 + 9(x-4)(52) + (x-4)(x-5)15 \\
 &\quad + (x-4)(x-5)(x-7)(1)
 \end{aligned}$$

$$f(8) = 448$$

⑤ Using Newton's divided difference formula
compute $f(5)$ from the data

x	1	2	4	7	12
y	22	30	82	106	216

Hints:

x	y	Δ	Δ^2	Δ^3	Δ^4
1	22	8			
2	30	26			
4	82	8	-3.6	0.194	-1.6
7	106	22	1.75	0.585	
12	216				

$$\begin{aligned}
 f(x) &= f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) \\
 &\quad + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) \\
 &\quad + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)
 \end{aligned}$$

$$f(5) = 102.144$$

- ⑥ Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data and hence find the value of y for $x=5$

x	4	6	8	10
y	1	3	8	16

x	y	Δ	Δ^2	Δ^3
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$u = \frac{x - x_0}{h} = \frac{5 - 4}{1} = 1$$

$$y(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots$$

$$f(5) = 29.125$$

⑦ Using Newton's forward interpolation formula, find the cubic polynomial which takes the values

x	0	1	2	3
$f(x)$	1	2	1	10

Hints

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1	1		
1	2	-1	-2	
2	1	9	10	12
3	10			

$$y(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$x_0 = 0$ $y_0 = 1$ $h = 1$ $u = \frac{x - x_0}{h} = x$

$$y(x) = 2x^3 - 7x^2 + 6x + 1$$

$$f(4) = 41$$

⑧ from the given data, find the number of students whose weight is between 60 and 70

Weight in lbs	0-40	40-60	60-80	80-100	100-120
No. of Students	250	120	100	70	50

x	y	cumulative frequency	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250	250				
60	120	370	120			
80	100	470	100	-20		
100	70	540	70	-30	-10	
120	50	590	50	-20	10	20

$$y(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0$$

$$u = \frac{x - x_0}{p} = \frac{70 - 40}{20} = 1.5$$

$$y(70) = 423.5937$$

$$x = 60$$

$$u = \frac{60-40}{20} = 1$$

$$y(60) = 370$$

No. of students whose weight is between 60 & 70 = 54

⑨ fit a Natural cubic Spline to the following data

x :	1	2	3	4
y	1	2	5	11

Also compute $y(1.5)$ and $y'(3)$.

Hints:

Here $h=1$

Let $M_0 = M_3 = 0$

The cubic Spline in $x_{i-1} \leq x \leq x_i$ is given by we have

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} [y_{i-1} - 2y_i + y_{i+1}]$$

$$M_0 + 4M_1 + M_2 = 6(2) = 12$$

$$M_1 + 4M_2 + M_3 = 18$$

$$4M_1 + M_2 = 12 \quad \text{--- ①}$$

$$M_1 + 4M_2 = 18 \quad \text{--- ②}$$

Solve ① & ②

$$d_1 = \frac{-1+8}{1} = 7, \quad d_2 = \frac{18+1}{1} = 19 \quad t = \frac{x-x_{i-1}}{h_i} = 7$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 21 \\ 78 \\ 57 \end{pmatrix}$$

$$k_0 = 4 \quad k_1 = 13 \quad k_2 = 22$$

$$x_0 = 1 \quad h_1 = 1 \quad x_1 = 2$$

$$f_0 = -8 \quad f_1 = -1$$

$$S_i(x) = (1-t) f_{i-1} + t f_i + h_i t (1-t) [k_{i-1} - d_i] \\ (1-t) - (k_{i-1} - d_i) t$$

$$t = \frac{x-x_{i-1}}{h_i}$$

$$\text{put } i=1, \quad t = \frac{x-x_0}{h_1} = x-1$$

$$S_1(x) = 8x + (x-1)(-1) + (1)(x-1)(-x) \\ [(4-7)(-x) - (13-7)(x+1)]$$

$$S_1(x) = 3x^3 - 9x^2 + 13x - 15$$

$$S_1(x) = 9x^2 - 18x + 13$$

$$S_1(1.5) = -5.625$$

$$S_1(1.75) = -3.7343$$

$$S_1'(1) = -4$$