UNIT 1

Solution of Equations and Eigen value Problems

Part A

(1) Write down the order of convergence and the condition for convergence of fixed point iteration Method.

(i) order of Convergence is 1

(ii) Condition for the Convergence of iteration Method is $|g'(x)| < 1$.

(2) Find an iterative formula to find the reciprocal of a given number N $(N \neq 0)$

Let $x = \frac{1}{N}$

$$N = \frac{1}{x}$$

$$f(x) = \frac{1}{x} - N, \quad f'(x) = \frac{-1}{x^2}$$

WKT $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left[\frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^2}}\right]$$

$$= x_n + x_n^2\left[\frac{1}{x_n} - N\right]$$

$$= x_n + x_n - N x_n^2$$

$$= 2x_n - N x_n^2$$

$$= x_n\left[2 - N x_n\right]$$

(3) What is the use of Power Method?

To find the numerically largest Eigen value of a given Matrix

(4) find an iterative formula to find $\sqrt{N}$ Where N is a +ve number.

Let $x = \sqrt{N} \Rightarrow x^2 = N$

$x^2 - N = 0$

$f(x) = x^2 - N$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

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(5) Solve the equations $x + 2y = 1$ & $3x - 2y = 7$ by Gauss Elimination Method.

Soln

$$(A, B) = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & -2 & 7 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -8 & 4 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$-8x = 4$

$\boxed{x = -2}$

$x + 2y = 1$

$-2 + 2y = 1$

$2y = 1 + 2$

$\boxed{y = 3/2}$

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(6) Evaluate $\sqrt{15}$ using Newton-Raphsons formula.

Soln $N = 15$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

Let $x_0 = 3.5$

$$x_1 = \frac{x_0^2 + N}{2x_0} = \frac{3.5^2+15}{2(3.5)} = 3.893$$

$$x_2 = \frac{x_1^2+N}{2x_1} = \frac{(3.893)^2+15}{2(3.893)} = 3.873$$

$$x_3 = \frac{x_2^2+N}{2x_2} = \frac{(3.873)^2+15}{2(3.873)} = 3.873.$$

$\therefore \sqrt{15} = 3.873$

(7) Write the procedure involved in Gauss Jordan elimination Method.

Soln Consider the system of eqns $AX = B$

If A is a diagonal matrix the gn system reduces to

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This system reduces to the following n eqns

$a_{11} x_1 = b_1, \quad a_{22} x_2 = b_2, \ldots \quad a_{nn} x_n = b_n.$

Hence we get the solutions as

$$x_1 = \frac{b_1}{a_{11}}, \quad x_2 = \frac{b_2}{a_{22}} \quad \cdots \quad x_n = \frac{b_n}{a_{nn}}$$

The Method of obtaining the solution of the system of equations by reducing the Matrix A to a diagonal Matrix is known as Gauss Jordan Method.

(8) Write down the condition for convergence of Newton Raphson Method for $f(x) = 0$.

The order of convergence is 2

Condition for convergence is $|f(x) f''(x)| < |f'(x)|^2$

1 9 Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ by Gauss Jordan Method

Soln

$$(A/I) = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_1$$

$\therefore$ Inverse of A is $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

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(10) What is the Criterion for the convergence of Newton Raphson Method?

The sequence $x_1, x_2, \ldots$ converges to the exact value if $|\varphi'(x)| < 1$

(i.e) if $|f(x) f''(x)| < |f'(x)|^2$

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(11) Give two direct method to solve a system of linear equations.

1. Gauss Elimination Method
2. Gauss Jordan Method

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(12) Give two indirect methods to solve a system of linear equations.

1. Gauss Jacobi Method
2. Gauss Seidal Method.

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(13) What is the use of Power Method?

The Power Method is used to find the largest eigen value in Magnitude and Corresponding eigen

Unit - I

Solution of Equations and Eigen value Problems

Part-B

(1) find Newton's iterative formula to find the reciprocal of a given number N and hence find the value of $\frac{1}{19}$

Hints:

Let $x = \frac{1}{N}$ $\therefore N = \frac{1}{x}$

$f(x) = \frac{1}{x} - N$; $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n(2 - Nx_n)$

To find $\frac{1}{19}$

Take $x_0 = \frac{1}{20} = 0.05$

$x_1 = 0.0525$, $x_2 = 0.05263$, $x_3 = 0.05263$

$x_4 = 0.05263$

$\therefore \frac{1}{19} = 0.05263$

(2) find a positive root of the equation $x^3 - 2x - 5 = 0$ by the Method of fixed Point iteration

Hints:

The root lies between 2 & 3

$x = (2x^2+5)^{1/3}$

Take $x_0 = 2.5$

$x_1 = \phi(x_0) = (2(x_0)^2+5)^{1/3} = 2.5962$

$x_2 = 2.6438$

$\vdots$

$x_{12} = 2.6906$

$x_{13} = 2.6906$

$\therefore$ The root is 2.6906

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(3) find a positive root of the equation $\cos x - 3x + 1 = 0$ by the Method of fixed Point iteration.

Hints:

$f(x) = \cos x - 3x + 1 = 0$

$f(0) = +ve$

$f(\pi/2) = -ve$

$\therefore$ The root lies between 0 & $\pi/2$

$x = \frac{1}{3}(1+\cos x)$

Take $x_0 = 0.6$

$x_1 = \frac{1}{3}(1+\cos x_0) = 0.60845$

$\vdots$

$x_5 = 0.60710$

$x_6 = 0.60710$

$\therefore$ The root is 0.60710

(4) find the least positive root of $x^4 - x - 10 = 0$ correct to 2 decimal places using Newton Raphson Method.

Soln Gn $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = -10 = -ve$

$f(2) = 4 = +ve$

$\therefore$ The root lies between 1 & 2

Take $x_0 = 2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.871$

$x_2 = 1.856$, $x_3 = 1.856$

$\therefore$ The root is 1.856

(5) find a root of $x \log_{10} x - 1.2 = 0$ Newton's Method correct to 3 decimal places.

Hints:

$f(x) = x \log_{10} x - 1.2$

$f(2) = -ve$

$f(3) = +ve$

$\therefore$ The root lies between 2 and 3

Take $x_0 = 2.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_1 = 2.7465$

$x_2 = 2.74065$

$x_3 = 2.74065$

$\therefore$ The root is 2.74065

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(6) find the approximate root of $xe^x = 3$ by Newton Raphson method correct to 3 decimal places.

Soln

$f(x) = xe^x - 3$

$f'(x) = xe^x + e^x$.

$f(1) = -0.2817 = -ve$

$f(2) = 11.4781 = +ve$

The root lies between 1 & 2

$|f(1)| < |f(2)|$

Take $x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_1 = 1.0518 \quad x_2 = 1.0499 \quad x_3 = 1.0499$

$\therefore$ The root is 1.0499

(7) Solve by Gauss elimination Method.

Hints:

$$[A, B] = \begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$= \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$2x + 3y - z = 5$

$-2y - z = -7$

$6z = 18$

$z = 3 \quad y = 2 \quad x = 1$

(8) Solve by Gauss Jacobi Method

$27x + 16y - z = 85$, $x + y + 54z = 110$

$6x + 5y + 2z = 72$

Hints:

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{27}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

Let $x_0 = 0$ $y_0 = 0$ $z_0 = 0$

$x_1 = 3.148$ $y_1 = 4.8$ $z_1 = 2.037$

$x_2 = 2.157$ $y_2 = 3.629$ $z_2 = 1.890$

$x_3 = 2.492$ $y_3 = 3.685$ $z_3 = 1.937$

$\therefore$ $x = 2.426$ $y = 3.573$ $z = 1.926$

(9) Apply Gauss Jordan Method to find the solution of the following eqn $x + 3y + 3z = 16$, $x + 4y + 3z = 18$

$x + 3y + 4z = 19$.

Hints:

$$(A, B) = \begin{bmatrix} 1 & 3 & 3 & 6 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 - 3R_3$$

$x = 1 \quad y = 2 \quad z = 3$

(10) Solve the following system by Gauss Seidal Method $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.

Soln

$x = \frac{1}{20}(17 - y + 2z)$, $y = \frac{1}{20}(-18 - 3x + z)$

$z = \frac{1}{20}(25 - 2x + 3y)$

$y = 0, \; z = 0$

$x_1 = 0.8500 \quad y_1 = -1.0275 \quad z_1 = 1.0109$

$x_2 = 1.0025 \quad y_2 = -0.9998 \quad z_2 = 0.9998$

$x_3 = 1.000 \quad y_3 = -1.000 \quad z_3 = 1.000$

$x = 1, \; y = -1, \; z = 1$

(11) Using Gauss Jordan Method find the Inverse of the Matrix $A = \begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{pmatrix}$

Hints :

$$[A/I] = A = \left[\begin{array}{ccc|ccc} 2 & 2 & 6 & 1 & 0 & 0 \\ 2 & 6 & -6 & 0 & 1 & 0 \\ 4 & -8 & -8 & 0 & 0 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/2 & 0 & 0 \\ 2 & 3 & -3 & 0 & 1/2 & 0 \\ 1 & -2 & -2 & 0 & 0 & 1/4 \end{array}\right] \begin{array}{l} R_1 \to R_1/2 \\ R_2 \to R_3/2 \\ R_3 \to R_3/4 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/2 & 0 & 0 \\ 0 & 2 & -6 & -1/2 & 1/2 & 0 \\ 0 & -3 & -5 & -1/2 & 0 & 1/4 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R, \\ R_3 \to R_3 - R, \end{array}$$

$$= \left[\begin{array}{cccccc} 1 & 0 & 6 & 3/4 & -1/4 & 0 \\ 0 & 1 & -3 & -1/4 & 1/4 & 0 \\ 0 & 0 & 1 & 5/56 & -3/56 & -1/56 \end{array}\right] R_3 \to R_3\left(\frac{3}{14}\right)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/14 & 1/14 & 6/56 \\ 0 & 1 & 0 & 1/56 & 5/56 & -3/56 \\ 0 & 0 & 1 & 5/56 & -3/56 & -1/56 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$$A^{-1} = \frac{1}{56}\begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix}$$

(12) Using Jacobi method, find the eigen values of $A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$

Hints :

$$\cot 2\theta = \alpha = \frac{a_{11} - a_{22}}{2a_{12}} = \frac{1-4}{6} = \frac{-1}{2}$$

$$\cot\theta = \beta = \alpha \pm \sqrt{1+\alpha^2} = -\tfrac{1}{2} \pm \sqrt{1+1/4}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

take $\beta = \frac{\sqrt{5}-1}{2} > 0$ $\sin\theta = \frac{1}{\sqrt{1+\beta^2}}$

$= \frac{2}{\sqrt{10+2\sqrt{5}}}$ and $\cos\theta = \frac{\beta}{\sqrt{1+\beta^2}} = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$

$b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta = \frac{5+3\sqrt{5}}{2}$

$b_{22} = a_{11} + a_{22} - b_{11} = \frac{5-3\sqrt{5}}{2}$

$\therefore$ eigen values are $\frac{5+3\sqrt{3}}{2}$ & $\frac{5-3\sqrt{5}}{2}$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\therefore X_1 = \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix} \text{ \& } X_2 = \begin{pmatrix} -2 \\ \sqrt{5}-1 \end{pmatrix}$$

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(13) find the largest eigen value and eigen vector of $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$ by using Power Method

Soln

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 0.857 \\ 0.285 \\ 1 \end{pmatrix} = 7X_2$$

$AX_2 = AX_2' = 7X_3'$

$AX_3 = AX_3' = 7X_4'$

$AX_4 = 7X_5'$

$\vdots$

$AX_{10} = \begin{pmatrix} 3.924 \\ 1.272 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 0.560 \\ 0.182 \\ 1 \end{pmatrix}$

$\therefore$ eigen Vector is $\begin{pmatrix} 0.560 \\ 0.182 \\ 1 \end{pmatrix}$

The eigen Value is 7