UNIT V GENERATING FUNCTIONS

Define Generating function.

A **generating function** describes an infinite sequence of numbers (a_n) by treating them like the coefficients of a series expansion. The sum of this infinite series is the generating function. Unlike an ordinary series, this formal series is allowed to diverge, meaning that the generating function is not always a true function and the "variable" is actually an indeterminate.

The generating function for 1, 1, 1, 1, 1, 1, 1, 1, 1, ..., whose ordinary generating function is

$$\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x}$$

The generating function for the geometric sequence 1, a, a^2, a^3, ... for any constant a:

$$\sum_{n=0}^{\infty}(ax)^n = \frac{1}{1-ax}$$

What is Partitions of integer?

Partitioning a positive n into positive summands and seeking the number of such partitions without regard to order is called Partitions of integer.

This number is denoted by $p(n)$. For example

P(1) = 1: 1
P(2) = 2: 2 = 1 + 1
P(3) = 3: 3 = 2 +1 = 1 + 1 +1
P(4) = 5: 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1
P(5) = 7: 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1+ 1 = 1 + 1 + 1 + 1 + 1

Define Exponential generating function

For a sequence a_0, a_1, a_2, a_3, … of real numbers.

$$f(x) = a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i\frac{x^i}{i!}$$

is called the exponential generating function for the given sequence.

Define Maclaurin series expansion of e^x and e^{-x}.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Adding these two series together, we get,

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Define Summation operator

Generating function for a sequence a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $a_0 + a_1 + a_2 + a_3$, … .

For $(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, consider the function $f(x)/(1-x)$

$$\frac{f(x)}{1-x} = f(x).\frac{1}{1-x} = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots$$

So $f(x)/(1-x)$ generates the sequence of sums a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $a_0 + a_1 + a_2 + a_3$,

$1/(1-x)$ is called the summation operator.

Recurrence relations

A **recurrence relation** is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

The term **difference equation** sometimes (and for the purposes of this article) refers to a specific type of recurrence relation. However, "difference equation" is frequently used to refer to *any* recurrence relation.

Fibonacci numbers

The recurrence satisfied by the Fibonacci numbers is the archetype of a homogeneous linear recurrence relation with constant coefficients (see below). The Fibonacci sequence is defined using the recurrence

$F_n = F_{n-1} + F_{n-2}$

with seed values $F_0 = 0$ and $F_1 = 1$

We obtain the sequence of Fibonacci numbers, which begins

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

First order linear recurrence relation

The general form of First order linear homogeneous recurrence relation can be written as

$a_{n+1} = d\,a_n$, $n \geq 0$, where d is a constant. The relation is first order since a_{n+1} depends on a_n.

a_0 *or* a_1 are called boundary conditions.

Second order recurrence relation

Let $k \in \mathbf{Z}^+$ and $C_0\ (\neq 0)$, $C_1, C_2, \ldots, C_k\ (\neq 0)$ be real numbers. If a_n, for $n \geq 0$, is a discrete function, then

$$C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} = f(n), \qquad n \geq k,$$

is a linear recurrence relation (with constant coefficients) of *order* k. When $f(n) = 0$ for all $n \geq 0$, the relation is called *homogeneous*; otherwise, it is called *nonhomogeneous*.

In this section we shall concentrate on the homogeneous relation of order two:

$$C_0a_n + C_1a_{n-1} + C_2a_{n-2} = 0, \qquad n \geq 2.$$

On the basis of our work in Section 10.1, we seek a solution of the form $a_n = cr^n$, where $c \neq 0$ and $r \neq 0$.

Substituting $a_n = cr^n$ into $C_0a_n + C_1a_{n-1} + C_2a_{n-2} = 0$, we obtain

$$C_0cr^n + C_1cr^{n-1} + C_2cr^{n-2} = 0.$$

With $c, r \neq 0$, this becomes $C_0r^2 + C_1r + C_2 = 0$, a quadratic equation which is called the *characteristic equation*. The roots r_1, r_2 of this equation determine the following three cases: (a) r_1, r_2 are distinct real numbers; (b) r_1, r_2 form a complex conjugate pair; or (c) r_1, r_2 are real, but $r_1 = r_2$. In all cases, r_1 and r_2 are called the *characteristic roots*.

Non-homogeneous recurrence relations

We now turn to the recurrence relations

$$a_n + C_1 a_{n-1} = f(n), \qquad n \geq 1, \tag{1}$$

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n), \qquad n \geq 2, \tag{2}$$

where C_1 and C_2 are constants, $C_1 \neq 0$ in Eq. (1), $C_2 \neq 0$, and $f(n)$ is not identically 0. Although there is no general method for solving all nonhomogeneous relations, for certain functions $f(n)$ we shall find a successful technique.

We start with the special case for Eq. (1), when $C_1 = -1$. For the nonhomogeneous relation $a_n - a_{n-1} = f(n)$, we have

$$\begin{aligned}
a_1 &= a_0 + f(1)\\
a_2 &= a_1 + f(2) = a_0 + f(1) + f(2)\\
a_3 &= a_2 + f(3) = a_0 + f(1) + f(2) + f(3)\\
&\vdots\\
a_n &= a_{n-1} + f(n) = a_0 + f(1) + \cdots + f(n) = a_0 + \sum_{i=1}^{n} f(i).
\end{aligned}$$

We can solve this type of relation in terms of n, if we can find a suitable summation formula for $\sum_{i=1}^{n} f(i)$.

UNIT- V : GENERATING FUNCTIONS

1.i) Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ given that $a_0=5$, $a_1=9$ and $a_2=15$.

ii)Solve the recurrence relation $S(n) = S(n-1)+2(n-1)$, with $S(0)=3$ $S(1)=1$, by finding its generating function.

2. i) A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the n^{th} month.

1) Give recurrence relation for the number of cars produced in the first n months by this factory.

2) How many cars are produced in the first year?

ii)Find the generating function of Fibonacci sequence

3. i) Find the generating functions for the following:
   a. 1,-1,1,-1,1-1,….. b. 1,0,1,0,1,0,…..

   ii) Determine the sequence for the following:
   a. $f(x)=(2x-3)^3$ b. $f(x)=1/(3-x)$

4. i) Find the coefficient of x^{15} in each of the following:
   a. $x^3(1-2x)^{10}$ b. $(x^3-5x)/(1-x)^3$

   ii) Analyze and solve the following
   a) Find the coefficient of x^{50} in $(x^7+x^8+x^9+\ldots)^6$
   b) Find the coefficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$

5. Analyze and solve the following

i) A company hires 11 new employees, each of whom is to be assigned to one of four subdivisions. Each subdivision will get at least one new employee. In how many ways can these assignments be made?

ii) Determine and give the sequence by each of the following exponential generating functions:
   a. $f(x)=3e^{3x}$ b. $f(x)=1/(1-x)$

6. i) Find the exponential generating function for the number of ways to arrange *n* letters, $n\geq 0$, selected from each of the following words. a. HAWAII b. ISOMORPHISM

   ii) Find the exponential generating functions for the following:
   a. 1,-1,1,-1,1-1,…..
   b. $1,2,2^2,2^3,2^4,\ldots$
   c. 0!,1!,2!,3!,……

7. i) Analyze and Solve the recurrence relation for the following:
   a. $a_n= 2(a_{n-1} - a_{n-2})$, where $n \geq 2$ and $a_0 =1$, $a_1 =2$.
   b. $a_n= 5a_{n-1} +6a_{n-2})$, where $n \geq 2$ and $a_0 =1$, $a_1 =3$.

   ii) Find a recurrence relation for the number of binary sequences of length *n* that have no consecutive 0's.

8.i) Find the recurrence relation, with initial condition, that uniquely determines each of the following geometric progressions.
   a. 2,10,50,250,…. b. 6,-18, 54,-162,…

   ii) Analyze and Solve the recurrence relation for fibonocci series.

9. Find the generating functions for the sequences

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9. Find the generating functions for the sequences

a. 1,2,3,3,3,.....;
b. 1,2,3,4,4,4,......
c. 0,1,0,0,0,.....
d. 0,1,2,3,4,......

10. i) Analyze and Solve the recurrence relation

$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

ii) Determine $(1 + \sqrt{3}i)^{10}$.

11. i) Develop the solution for the recurrence relation

$a_n - 3a_{n-1} = 5(3^n)$, where $n \geq 1$, $a_0 = 2$.

ii) Develop the solution for the recurrence relation

$a_n - a_{n-1} = 3n^2$, where $n \geq 0$, $a_0 = 7$.

12. Solve the following recurrence relation by the method of generating functions:

a. $a_{n+1} - a_n = 3^n$, $n \geq 0$ and $a_0 = 1$

b. $a_{n+1} - a_n = n^2$, $n \geq 0$ and $a_0 = 1$

c. $a_{n+2} - 3a_{n+1} + 2a_n = 0$, $n \geq 0$ and $a_0 = 1$, $a_1 = 6$

d. $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0$ and $a_0 = 1$, $a_1 = 2$

13. i) Give the solution for the following recurrence relations

a. $a_{n+1} - a_n = 2n + 3$, $n \geq 0$, $a_0 = 1$

b. $a_{n+1} - a_n = 3n^2 - n$, $n \geq 0$ and $a_0 = 3$

ii) Give the solution for the following recurrence relations

a. $a_{n+1} - 2a_n = 5$, $n \geq 0$, $a_0 = 1$

b. $a_{n+1} - 2a_n = 2^n$, $n \geq 0$ and $a_0 = 1$

14. i) Give the solution for the following systems of recurrence relations.

$a_{n+1} = -2a_n - 4b_n$

$b_{n+1} = 4a_n + 6b_n$, $n \geq 0$, $a_0 = 1$ $b_0 = 0$

ii) Give the solution for the following systems of recurrence relations.

$a_{n+1} = 2a_n - b_n + 2$

$b_{n+1} = -a_n + 2b_n - 1$, $n \geq 0$, $a_0 = 0$ $b_0 = 1$