UNIT III MATRICES, COLOURING AND DIRECTED GRAPH

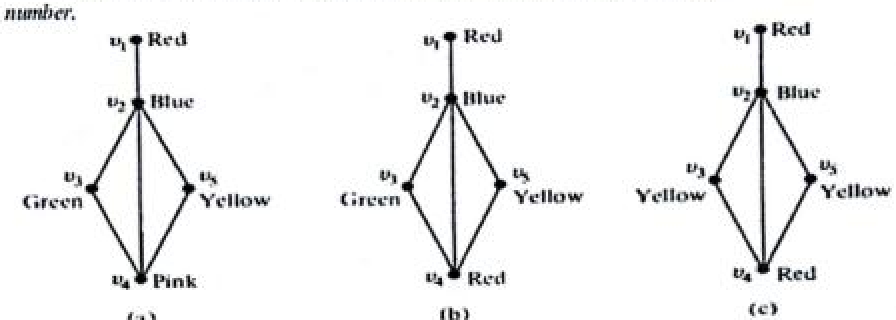
1. What is proper coloring?

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* (simply *coloring*) of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a *properly colored graph*.

2. Define Chromatic number

A graph G that requires k different colors for its proper coloring, and no less, is called *k-chromatic* graph, and the number k is called the *chromatic number* of G.

The minimum number of colors required for the proper coloring of a graph is called *Chromatic number*.



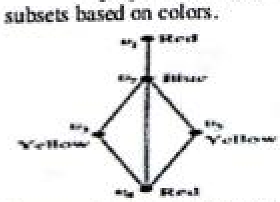
The above graph initially colored with 5 different colors, then 4, and finally 3. So the chromatic number is 3. i.e., The graph is 3-chromatic.

3. Write the properties of chromatic numbers (observations).

- A graph consisting of only isolated vertices is 1-chromatic.
- Every tree with two or more vertices is 2-chromatic.
- A graph with one or more vertices is at least 2-chromatic.
- A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.
- A complete graph consisting of n vertices is n-chromatic.

4. Define Chromatic partitioning

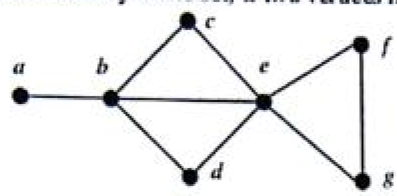
A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets based on colors.



For example, the coloring of the above graph produces the portioning $\{v_1, v_4\}$, $\{v_2\}$, and $\{v_3, v_5\}$.

5. Define independent set and maximal independent set.

A set of vertices in a graph is said to be an *independent set* of vertices or simply independent set (or an internally stable set) if two vertices in the set are adjacent.



For example, in the above graph produces {a, c, d} is an independent set.

A single vertex in any graph constitutes an independent set.

A maximal independent set is an independent set to which no other vertex can be added without destroying its independence property.

{a, c, d, f} is one of the maximal independent set. {b, f} is one of the maximal independent set.

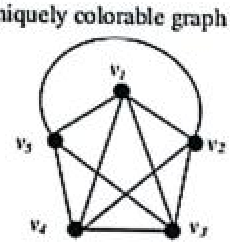
The number of vertices in the largest independent set of a graph G is called the *independence number* (or coefficients of internal stability), denoted by β(G).

For a K-chromatic graph of n vertices, the independence number $\beta(G) \geq \frac{n}{k}$.

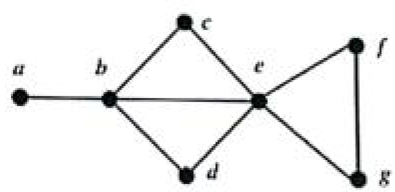
6. Define uniquely colorable graph.

A graph that has only one chromatic partition is called a uniquely colorable graph. For example,

Uniquely colorable graph G:

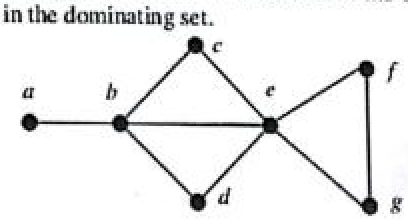


Not uniquely colorable graph H:



7. Define dominating set.

A dominating set (or an externally stable set) in a graph G is a set of vertices that dominates every vertex v in G in the following sense: Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set.



{b, g} is a dominating set, {a, b, c, d, f} is a dominating set. A is a dominating set need not be independent set. Set of all vertices is a dominating set.

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

{b, e} is a minimal dominating set.

8. Define Chromatic polynomial.

A graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of polynomial. This polynomial is called the Chromatic polynomial of G.

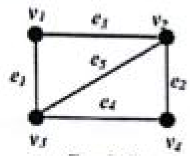
The value of the Chromatic polynomial $P_n(\lambda)$ of a graph with n vertices the number of ways of properly coloring the graph , using λ or fewer colors.

9. Define Matching (Assignment).

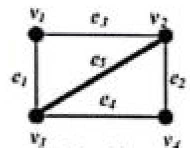
A *matching* in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is a matching.

A *maximal matching* is a matching to which no edge in the graph can be added.

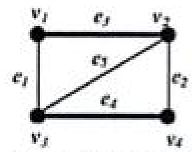
The maximal matching with the largest number of edges are called the *largest maximal matching*.



Graph G



Matching



Maximal matching

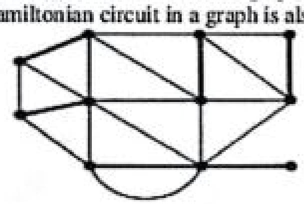
10. What is Covering?

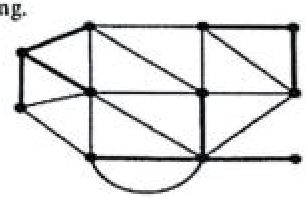
A set g of edges in a graph G is said to be cover og G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be a covering (or an edge covering, or a coverring subgraph) of G.

Every graph is its own covering.

A spanning tree in a connected graph is a covering.

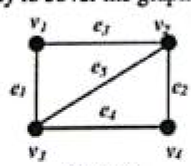
A Hamiltonian circuit in a graph is also a covering.



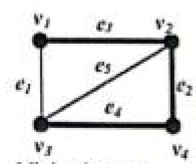


11. Define minimal cover.

A **minimal covering** is a covering from which no edge can be removed without destroying it ability to cover the graph G.



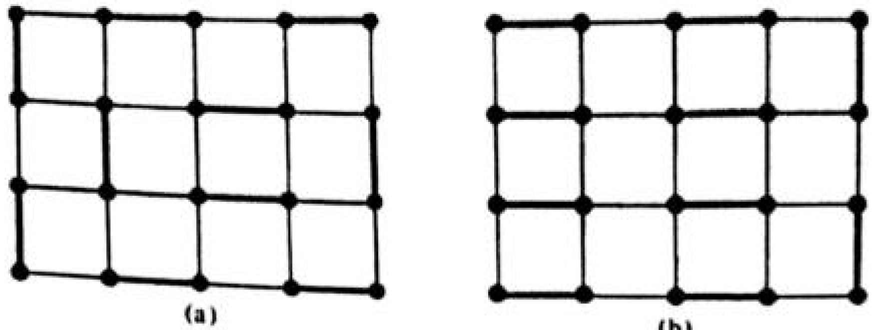
Graph G



Minimal cover

12. What is dimer covering?

A covering in which every vertex is of degree one is called a *dimer covering* or a *1-factor*. A dimmer covering is a maximal matching because no two edges in it are adjacent. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



Two dimmer coverings.

13. Define four color problem / conjecture.

- Every planar graph has a chromatic number of four or less.
- Every triangular planar graph has a chromatic number of four or less.
- The regions of every planar, regular graph of degree three can be colored properly with four colors.

14. State five color theorem

Every planar map can be properly colored with five colors.
i.e., the vertices of every plannar graph can be properly colored with five colors.

15. Write about vertex coloring and region coloring.

A graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G* and vice versa.

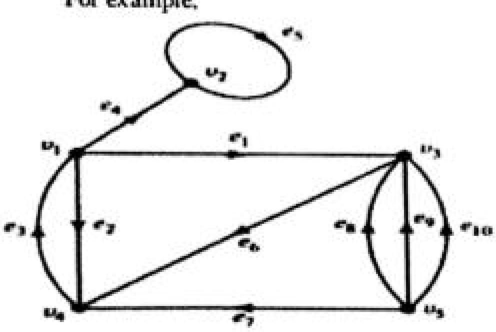
What is meant by regularization of a planar graph?

- Remove every vertex of degree one from the graph G does not affect the regions of a plannar graph.
- Remove every vertex of degree two and merge the two edges in series from the graph G.
- Such a transformation may be called regularization of a planar graph.

16. Directed graphs

A *directed graph* (or a *digraph, or an oriented graph*) G consists of a set of vertices $V = \{ v_1, v_2, \ldots \}$, a set of edges $E = \{ e_1, e_2, \ldots \}$, and a mapping Ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

For example,

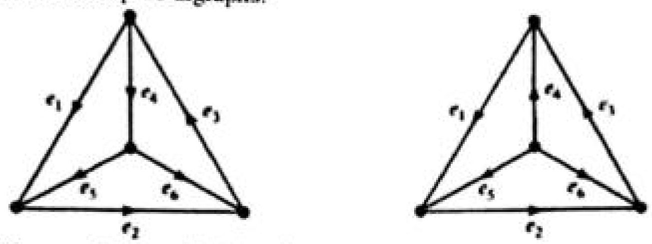


17. Define isomorphic digraph.

Among directed graphs, if their labels are removed, two isomorphic graphs are indistinguishable then these graphs are **isomorphic digraph**.

For example,

Two isomorphic digraphs.



Two non-isomorphic digraphs.

18. List out some types of directed graphs

- Simple Digraphs
- Asymmetric Digraphs (Anti-symmetric)
- Symmetric Digraphs
- Simple Symmetric Digraphs
- Simple Asymmetric Digraphs
- Complete Digraphs
- Complete Symmetric Digraphs
- Complete Asymmetric Digraphs (tournament)
- Balance digraph (a pseudo symmetric digraph or an isograph)

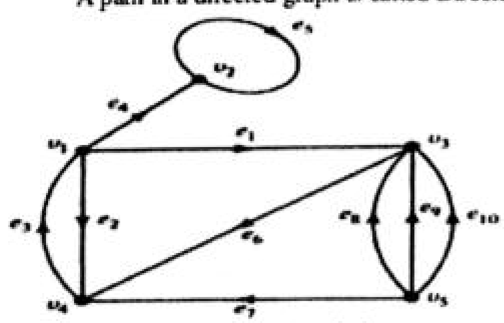
19. Define binary relations.

In a set of objects, X, where X={ x_1, x_2, ...}, A *binary relation R* between pairs (x_i, x_j) can be written as $x_i \, R \, x_j$ and say that x_i has relation R to x_j.

If the binary relation R is reflexive, symmetric, and transitive then R is an equivalence relation. This produces equivalence classes.

20. What is Directed path?

A path in a directed graph is called Directed path.



$v_5 \, e_8 \, v_3 \, e_6 \, v_4 \, e_3 \, v_1$ is a directed path from v_5 to v_1.

Whereas $v_5 \, e_7 \, v_4 \, e_6 \, v_3 \, e_1 \, v_1$ is a semi-path from v_5 to v.

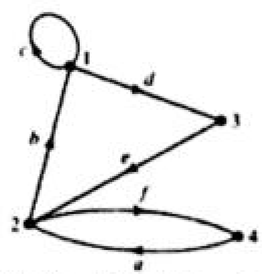
21. Write the types of connected digraphs

- **Strongly connected digraph:** A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.
- **Weakly connected digraph**: A digraph G is said to be weakly connected if its corresponding undirected graph is connected. But G is not strongly connected.

22. Define Euler digraphs

In a digraph G, a closed directed walk which traverses every edge of G exactly once is called a *directed Euler line*. A digraph containing a directed Euler line is called an **Euler digraphs**

For example,



It contains directed Euler line **a b c d e f.**

23. What is teleprinter's problem.

Constructing a longest circular sequence of 1's and 0's such that no subsequence of r bits appears more than once in the sequence.

Teleprinter's problem was solved in 1940 by I.G. Good using digraph.

UNIT III - MATRICES, COLOURING & DIRECTED GRAPH

1. i) Show that digraph representing the relation "congruent mod 3" on a set of finite integers 1-11 is an Equivalence graph.

ii) Derive the chromatic polynomial for the given graph and use that to find information on chromatic number of the graph

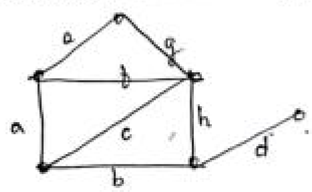
2. i) Create an algorithm to find all cycles in a digraph using exhaustive search method and state the precautions 10 to be taken.

ii) Define a k-chromatic graph? Prove that every tree with two or more vertices is 2-chromatic?

3. i) If A(G) is an incidence matrix of a connected graph G with n vertices then prove that rank of A(G) is n-1.

ii) Explain the theorem the reduced incidence matrix of a tree is non-singular.

iii) Generate the circuit matrix for the following graph



4. i) If the edges of a connected graph are arranged in the same order for the columns of the incidence matrix A and the path matrix P(x,y), then prove that the product (mod 2) $A.P^T(x,y) = M$, where the matrix M has 1's in two rows x and y, and the rest of the n-2 rows are all zeros.

ii)Explain the theorem Every tree with two or more vertices is 2-chromatic.

5. i) Define : 1. Adjacency matrix and 2. Incidence matrix of a graph with examples.

ii)Show that a connected multi-graph has an Euler Circuit if and only if each of its vertices has an even degree.

6. i) Show that a graph with atleast one edge is 2 chromatic if and only if it has no circuits of odd length.

ii) Show that d_{max} is the maximum degree of the vertices in a graph G.

7. i) What is an maximal independent set? State the procedure of finding all maximal independent sets.

ii) What is an minimal dominating set? State the procedure of finding all minimal dominating sets.

8. i) Prove that a covering *g* of a graph is minimal if and only if *g* contains no paths of length three or more.

ii) Define the incidence matrix, of a graph G? Prove that the rank of an incidence matrix of a connected graph with n vertices is n-1?

9. State and explain the Four-Color Problem.

10. i) Explain isomorphic digraphs with example.

ii) List and explain various types of digraphs with neat diagram.

11. i) How binary relations are closely related to theory of graphs? Explain in detail.

ii) Write short notes on condensation of digraphs.

12. Give the proof for the following theorem

A diagraph G is an Euler digraph if and only if G is connected and is balanced [ie. $d^-(v)=d^+(v)$ for every vertex v in G].

13. i) Explain the problem of partitioning in graph with example.

ii) Give the proof for the following theorem

m-vertex graph is a tree if its chromatic polynomial is Pm (n) = n (n-1) ^ (m-1)?

14. Describe chromatic number and chromatic polynomial of a graph? Determine the chromatic number and chromatic polynomial of the following graphs