UNIT II TREES, CONNECTIVITY & PLANARITY

1. Define Spanning trees.

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices (maximal tree subgraph).

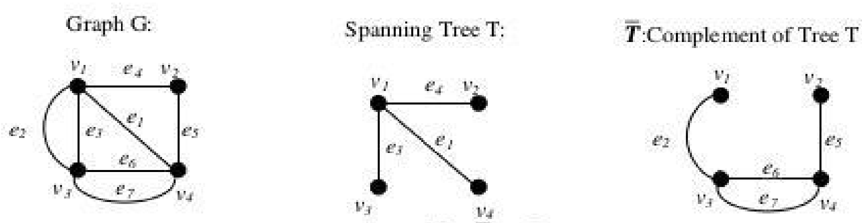
2. Define Branch and chord.

An edge in a spanning tree *T* is called a *branch* of *T*. An edge of G is not in a given spanning tree *T* is called a *chord* (*tie* or *link*).

Edge e_1 is a branch of *T* *Edge e_5 is a chord* of *T*

3. Define complement of tree.

If T is a spanning tree of graph G, then the complement of T of G denoted by $\overline{T}$ is the collection of chords. It also called as *chord set* (*tie* set or *cotree*) of T



$$T \cup \overline{T} = G$$

4. Define Rank and Nullity:

A graph G with n number of vertices, e number of edges, and k number of components with the following constraints $n - k \geq 0$ and $e - n + k \geq 0$.

Rank $r = n - k$

Nullity $\mu = e - n + k$ (Nullity also called as *Cyclomatic number* or *first betti number*)

Rank of G = number of branches in any spanning tree of G

Nullity of G = number of chords in G

Rank + Nullity = e = number of edges in G

5. How Fundamental circuits created?

Addition of an edge between any two vertices of a tree creates a circuit. This is because there already exists a path between any two vertices of a tree.

6. Define Spanning trees in a weighted graph

A spanning tree in a graph G is a minimal subgraph connecting all the vertices of G. If G is a weighted graph, then the weight of a spanning tree *T* of G is defined as the sum of the weights of all the branches in *T*.

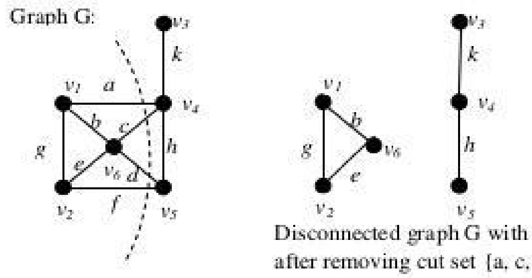
A spanning tree with the smallest weight in a weighted graph is called a *shortest spanning tree (shortest-distance spanning tree* or *minimal spanning tree)*.

7. Define degree-constrained shortest spanning tree.

A shortest spanning tree T for a weighted connected graph G with a constraint $d(v_i) \leq k$ for all vertices in T. for k=2, the tree will be Hamiltonian path.

8. Define cut sets and give example.

In a connected graph G, a cut-set is a set of edges whose removal from G leave the graph G disconnected.



Disconnected graph G with 2 components after removing cut set {a, c, d, f}

Possible cut sets are {a, c, d, f}, {a, b, e, f}, {a, b, g}, {d, h, f}, {k}, and so on.
{a, c, h, d} is not a cut set, because its proper subset {a, c, h} is a cut set.
{g, h} is not a cut set.

A minimal set of edges in a connected graph whose removal reduces the rank by one is called minimal cut set (simple cut-set or cocycle). Every edge of a tree is a cut set.

9. Write the Properties of cut set

- Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.
- In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.
- Every circuit has an even number of edges in common with any cut set.

10. Define Fundamental circuits

Adding just one edge to a spanning tree will create a cycle; such a cycle is called a **fundamental cycle (Fundamental circuits**). There is a distinct fundamental cycle for each edge; thus, there is a one-to-one correspondence between fundamental cycles and edges not in the spanning tree. For a connected graph with *V* vertices, any spanning tree will have $V - 1$ edges, and thus, a graph of *E* edges and one of its spanning trees will have $E - V + 1$ fundamental cycles.

11. Define Fundamental cut sets

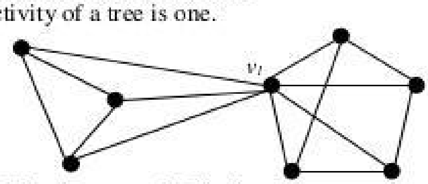
Dual to the notion of a fundamental cycle is the notion of a **fundamental cutset**. By deleting just one edge of the spanning tree, the vertices are partitioned into two disjoint sets. The fundamental cutset is defined as the set of edges that must be removed from the graph *G* to accomplish the same partition. Thus, each spanning tree defines a set of $V - 1$ fundamental cutsets, one for each edge of the spanning tree.

12. Define edge Connectivity.

Each cut-set of a connected graph G consists of certain number of edges. The number of edges in the smallest cut-set is defined as the **edge Connectivity of G.**

The **edge Connectivity** of a connected graph G is defined as the minimum number of edges whose removal reduces the rank of graph by one.

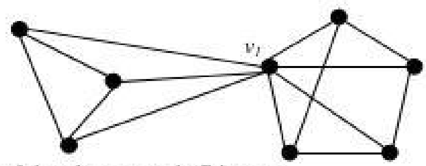
The edge Connectivity of a tree is one.



The edge Connectivity of the above graph G is three.

13. Define vertex Connectivity

The **vertex Connectivity** of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. The vertex Connectivity of a tree is one.

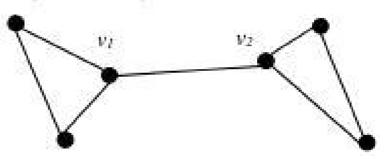


The vertex Connectivity of the above graph G is one.

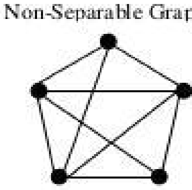
14. Define separable and non-separable graph.

A connected graph is said to be separable graph if its vertex connectivity is one. All other connected graphs are called non-separable graph.

Separable Graph G:

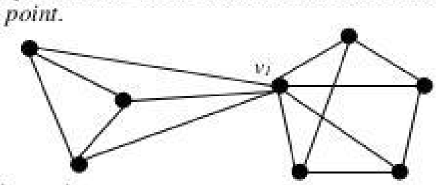


Non-Separable Graph H:



15. Define articulation point.

In a separable graph a vertex whose removal disconnects the graph is called *a cut-vertex, a cut-node, or an articulation point.*



v_1 is an articulation point.

16. What is Network flows

A **flow network** (also known as a transportation **network**) is a **graph** where each edge has a capacity and each edge receives a **flow**. The amount of **flow** on an edge cannot exceed the capacity of the edge.

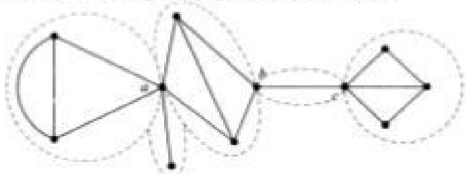
17. Define max-flow and min-cut theorem (equation).

The maximum flow between two vertices a and b in a flow network is equal to the minimum of the capacities of all cut-sets with respect to a and b.

The max. flow between two vertices = Min. of the capacities of all cut-sets.

18. Define component (or block) of graph.

A separable graph consists of two or more non separable subgraphs. Each of the largest nonseparable is called a block (or component).



The above graph has 5 blocks.

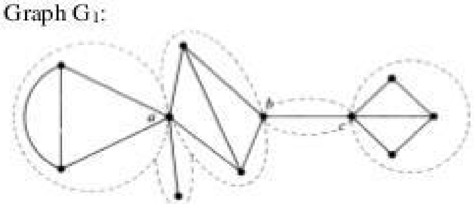
19. Define 1-Isomorphism

A graph G_1 was 1-Isomorphic to graph G_2 if the blocks of G_1 were isomorphic to the blocks of G_2.

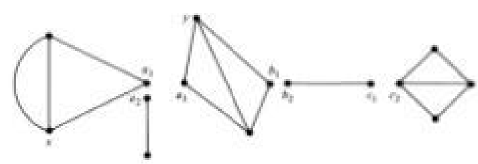
Two graphs G_1 and G_2 are said to be 1-Isomorphic if they become isomorphic to each other under repeated application of the following operation.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

Graph G_1:



Graph G_2:



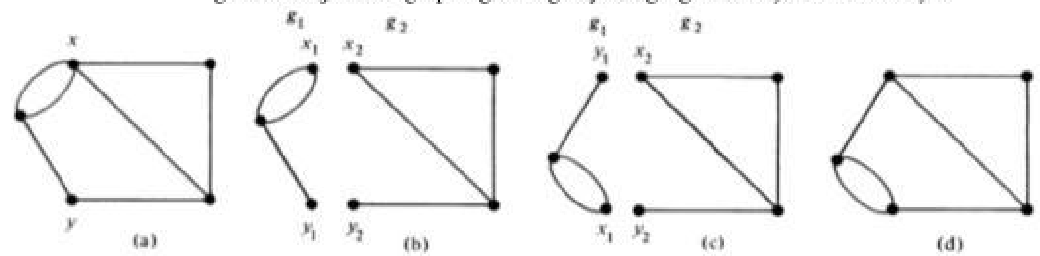
Graph G_1 is 1-Isomorphism with Graph G_2.

20. Define 2-Isomorphism

Two graphs G_1 and G_2 are said to be **2-Isomorphic** if they become isomorphic after undergoing *operation 1* or *operation 2*, or both operations any number of times.

Operation 1: "Split" a cut-vertex into two vertices to produce two disjoint subgraphs.

Operation 2: "Split" the vertex x into x_1 and x_2 and the vertex y into y_1 and y_2 such that G is split into g_1 and g_2. Let vertices x_1 and y_1 go with g_1 and vertices x_2 and y_2 go with g_2. Now rejoin the graphs g_1 and g_2 by merging x_1 with y_2 and x_2 with y_1.



21. Briefly explain Combinational and geometric graphs

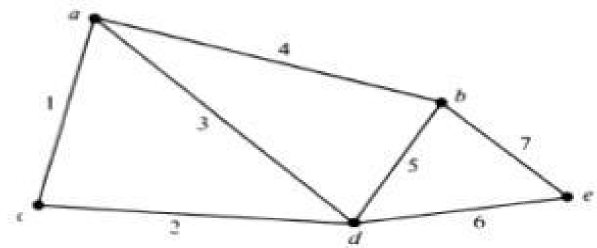
An abstract graph G can be defined as $G = (V, E, \Psi)$

Where the set V consists of five objects named *a*, *b*, *c*, *d*, and *e*, that is, $V = \{a, b, c, d, e\}$ and the set E consist of seven objects named 1, 2, 3, 4, 5, 6, and 7, that is, $E = \{1, 2, 3, 4, 5, 6, 7\}$, and the relationship between the two sets is defined by the mapping Ψ, which consist of

Ψ= [1→(a, c), 2→(c, d) , 3→(a, d) , 4→(a, b) , 5→(b, d) , 6→(d, e) , 7→(b, e)].

Here the symbol 1→*(a, c)*, says that object 1 from set E is mapped onto the pair (a, c) of objects from set V.

This combinatorial abstract object G can also be represented by means of a geometric figure.



The figure is one such geometric representation of this graph G.

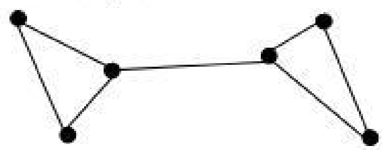
Any graph can be geometrically represented by means of such configuration in three dimensional Euclidian space.

22. Distinguish between Planar and non-planar graphs

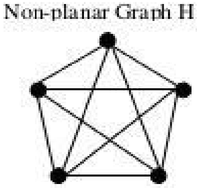
A graph G is said to be *planar* if there exists some geometric representation of G which can be drawn on a plan such that no two of its edges intersect.

A graph that cannot be drawn on a plan without crossover its edges is called *non-planar*.

Planar Graph G:



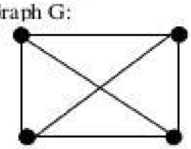
Non-planar Graph H:



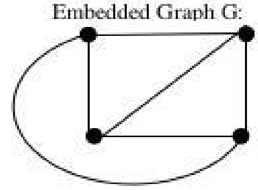
23. Define embedding graph.

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Graph G:

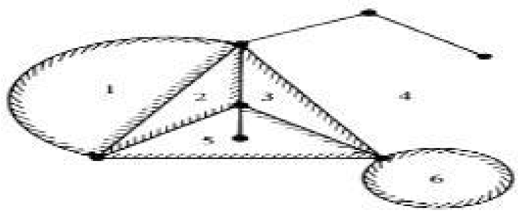


Embedded Graph G:



24. Define region in graph.

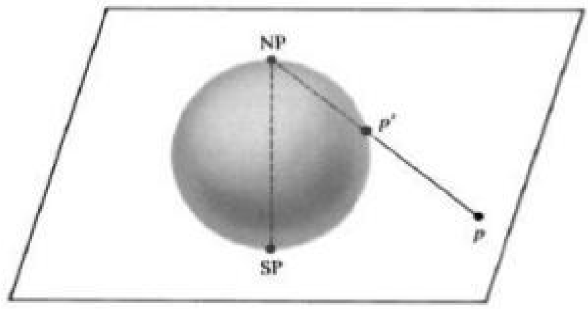
In any planar graph, drawn with no intersections, the edges divide the planes into different **regions (windows, faces, or meshes)**. The regions enclosed by the planar graph are called **interior faces** of the graph. The region surrounding the planar graph is called the **exterior** (or infinite or unbounded) face of the graph. Prepared by G. Appasami, Assistant professor, Dr. pauls Engineering College.



The graph has 6 regions.

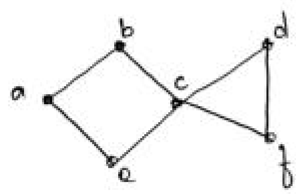
25. Why the graph is embedding on sphere.

To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of sphere. This is done by stereographic projection.

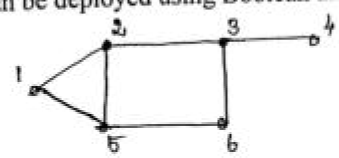


UNIT II - TREES, CONNECTIVITY & PLANARITY

1. Give the proof for the following theorems
   i) Every tree has either one or two centers.
   ii) A graph is a tree if and only if it is minimally connected.
   iii) Number of vertices in a binary tree is always odd.
   iv) Number of pendent vertices in a binary tree is (n+1)/2.
2. i) Explain the proof of the theorem
   Two graphs are 2-isomorphic if and only if they have circuit correspondence.
   ii) For the following graph, find the all maximal independent sets.



   iii) If the distance d(x,y) between two vertices x and y in a graph is defined to be the length of the shortest path connecting them, then prove that the distance function is a metric.
3. Explain with proof that a graph is non-planar if and only if it contains a sub-graph homomorphic to K_5 or $K_{3,3}$.
4. i) Establish and prove the relation between vertex connectivity, edge connectivity and number of vertices and edges.
   ii) Explain the proof of following theorem
   The largest number of edges in a planar graph is 3n-6, where n is the number of vertices in, the graph.
5. i) State and prove theorems relating fundamental circuits and fundamental cut set w.r.t a spanning tree.
   ii) Explain with proof that the number of labeled trees with n vertices is n^{n-2}.
6. i) In a network as shown below, during propagation of worms, protection strategy needs to be adopted against the attack. Find the optimal number of nodes where the defense strategy can be deployed using Boolean arithmetic.



   ii) Given a connected graph G, find the rank of a matrix that defines the graph within

2- isomorphism.

7. i) Prove that an Euler graph cannot have a cut-set with odd number of edges.

ii) Develop the algorithm to find cut vertices and bridges in a graph.

8. i) Give the proof of the following theorem

A spanning tree T of a given weighted connected graph G, is a shortest spanning tree of G if and only if there exists no other spanning tree at a distance of one from T whose weight is smaller than that of T.

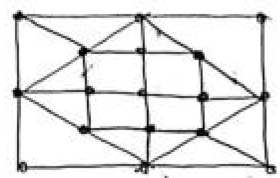
ii) Create the algorithm to find the shortest path from a given source vertex to any vertex in a graph with an example.

9. i) Explain the steps involved in testing for planarity of graphs.

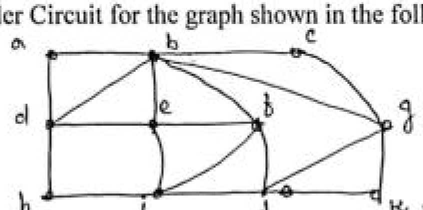
ii) Define graph isomorphism and characterize graphs possessing 1-isomorphism and 2-isomorphism.

10. i) Examine if the following graphs are planar or non-planar i) K_4, ii) $K_{3,3}$. Give reason.

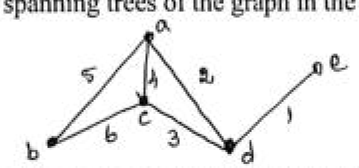
ii) Find the number of vertices, edges and regions for the following planar graph and verify that Euler's Theorem for connected planar graphs is satisfied



11. i) Find an Euler Circuit for the graph shown in the following figure.

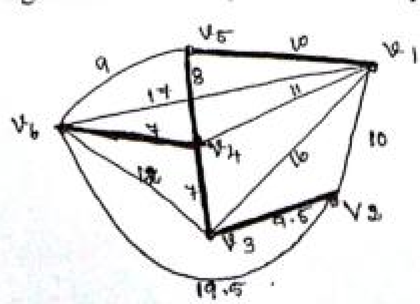


ii) Predict all spanning trees of the graph in the following figure.



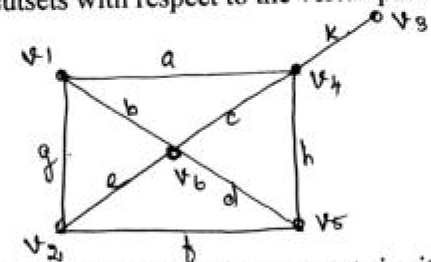
12. i) Show that when any edge is removed from K_5, the resulting subgraph is planar. Is this true for the graph $K_{3,3}$

ii) Using the algorithm of Kruskal, find a shortest spanning tree in the following graph



13. i) Show that an Hamiltonian Path is a spanning tree.

ii) List all cutsets with respect to the vertex pair v2,v3 in the graph.



14. i) Prove that the ring sum of two cut sets is either a third cut set or an edge disjoint union of two cut sets.

ii) Show that the edge connectivity and vertex connectivity of the graph in the following figure are each equal to three.