UNIT III GRAPH THEORY

PART – A

1. **Define Graph.**

Ans: A graph G = (V,E) consists of a finite non empty set V, the element of which are the vertices of G, and a finite set E of unordered pairs of distinct elements of V called the edges of G.

2. **Define complete graph.**

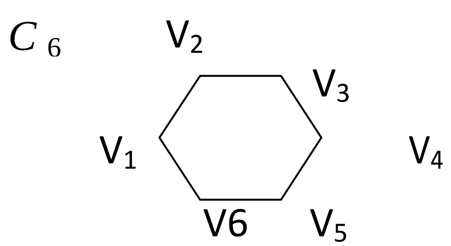
Ans: A graph of n vertices having each pair of distinct vertices joined by an edge is called a Complete graph and is denoted by K_n.

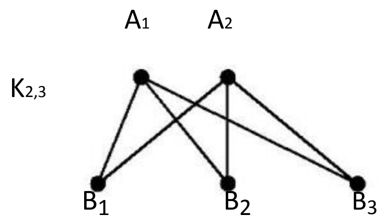
3. **Define regular graph.**

Ans: A graph in which each vertex has the same degree is called a regular graph. A regular graph has k – regular if each vertex has degree k.

4. **Define Bipartite Graph with example.**

Ans: Let G = (V,E) be a graph. G is bipartite graph if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 , called a bipartition, such that each edge has one end in V_1 and in V_2 . For eg

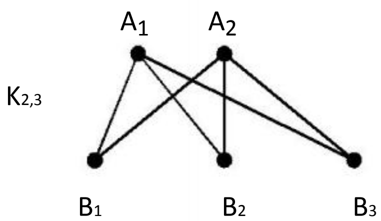


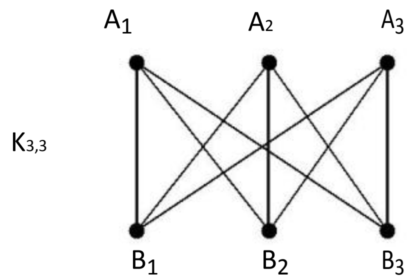
5.	**Define complete bipartite graph with example**
	Ans: A complete bipartite graph is a bipartite graph with bipartition V_1 and V_2 in which each vertex of V_1 is joined by an edge to each vertex of V_2. For eg. 
6.	**Define Subgraph.**
	Ans: A graph H = (V_1,E_1) is a subgraph of G = (V,E) provided that V_1 ,E_1 and for each e $\in E_1$, both ends of e are in V_1.
7.	**Define Isomorphism of two graphs.**
	Ans: Two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are the same or isomorphic, if there is a bijection F :$V_1 \rightarrow V_2$ such that (u,v) $\in E_1$ if and only if (F(u), F(v)) $\in E_2$.
8.	**Define strongly connected graph.**
	Ans: A digraph G is said to be strongly connected if for every pair of vertices, both vertices of the pair are reachable from one another.
9.	**State the necessary and sufficient conditions for the existence of an Eulerian path in a connected graph.**
	Ans: A connected graph contains an Euler path if and only if it has exactly two vertices of odd degree.
10.	**State Handshaking theorem.**
	Ans: If G = (V, E) is an undirected graph with *e* edges, then $\sum_i \deg(v_i) = 2e$
11.	**Define adjacency matrix.**
	Ans: Let G = (V,E) be a graph with n vertices . An “n *x* n” matrix A is an adjacency matrix for G if and only if for $\le$, $\le$, $A(i,j) = \begin{cases} 1 & for\ (i,j)\ in\ E \\ 0 & for\ (i,j)\ is\ not\ in\ E \end{cases}$
12.	**Define Connected graph.**
	Ans: A graph for which each pair of vertices is joined by a trail is connected.
13.	**Define Pseudo-graph.**
	Ans: A graph is called a pseudo-graph if it has both parallel edges and self loops.
14.	**Does there exist a simple graph with five vertices of the 0, 1, 2, 2, 3 degrees? If so, draw such a graph.**
	Ans:

Yes.

15. **Draw a complete bipartite graph of $K_{2,3}$ and $K_{3,3}$**

Ans:





16. **Define spanning subgraph.**

Ans: Let a graph H = (V_1,E_1) is a subgraph of G = (V,E). H is a spanning subgraph of G if H is a subgraph of G with V_1= V and $E_1 \subset$ E.

17. **Define Induced subgraph.**

Ans: A graph H = (V_1,E_1) is a subgraph of G = (V,E). H is an induced subgraph of G such that E_1 consists of all the edges of G with both ends in V_1.

18. **Define Eulerian Circuit.**

Ans: A circuit in a graph that includes each edge exactly once, the circuit is called an Eulerian circuit.

19. **State the condition for Eulerian cycle.**

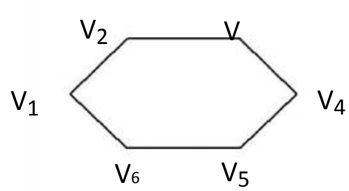
Ans: (i) Starting and ending pts are same.

(ii) Cycle should contain all edges of graph but exactly once

20 **Show that C_6 is a bipartite graph?**

Ans:

C_6 vertex set is partitioned into two set $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{ v_2, v_4, v_6\}$, where every edge of C_6 joins a vertex in V_1 to a vertex in V



PART - B

1(a) **State and prove Handshaking Theorem.**

If G = (V, E) is an undirected graph with *e* edges, then $\sum_i \deg(v_i) = 2e$

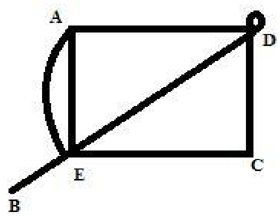
Proof: Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

Therefore, all the *e* edges contribute (2*e*) to the sum of the degrees of the vertices.

Hence $\sum_i \deg(v_i) = 2e$.

1(b)	**In any graph show that the number of odd vertices is even.** Let G = (V, E) be the undirected graph. Let V_1 and V_2 be the set of vertices of G of even and odd degrees respectively. Then by hand shaking theorem, $2e = \sum_{v_i \in v_1} deg(v_i) + \sum_{v_j \in v_2} deg(v_j)$. Since each deg($v_i$) is even, $\sum_{v_i \in v_1} deg(v_i)$ is even. Since LHS is even, we get $\sum_{v_j \in v_2} deg(v_j)$ is even. Since each deg(v_j) is odd, the number of terms contain in $\sum_{v_j \in v_2} deg(v_j)$ or v_2 is even, that is, the number of vertices of odd degree is even.
2(a)	**Prove that a simple graph with at least two vertices has at least two vertices of same degree.** **Proof:** Let G be a simple graph with n $\geq$ 2 vertices. The graph G has no loop and parallel edges. Hence the degree of each vertex is $\leq$ n-1. Suppose that all the vertices of G are of different degrees. Following degrees 0, 1, 2, …, n-1 are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree n-1 then v has n-1 adjacent vertices. Because v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction proves that simple graph contains two vertices of same degree.
2(b)	**Prove that the maximum number of edges in a simple graph with n vertices is** $nc_2 = \frac{n(n-1)}{2}$ **Proof:** We prove this theorem, by the method of mathematical induction. For n = 1, a graph with 1 vertex has no edges. Therefore the result is true for n = 1. For n = 2, a graph with two vertices may have atmost one edge. Therefore 2 (2 –1) / 2 = 1. Hence for n = 2, the result is true. Assume that the result is true for n = k, i.e, a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges. Then for n = k + 1, let G be a graph having n vertices and G′ be the graph obtained from G, by deleting one vertex say, „v" $\in$ V(G). Since G′ has k vertices then by the hypothesis, G′ has atmost $\frac{k(k-1)}{2}$ edges. Now add the vertex v to G′. „v" may be adjacent to all the k vertices of G′. Therefore the total number of edges in G are $\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$. Therefore the result is true for n = k+1. Hence, the maximum number of edges in a simple graph with „n" vertices is $\frac{n(n-1)}{2}$.
3(a)	**Show that a simple graph G with *n* vertices is connected if it has more than** $\frac{(n-1)(n-2)}{2}$ ***edges*** **Proof:** Suppose G is not connected. Then it has a component of k vertices for some k, The most edges G could have is

	$C(k,2)+C(n-k,2)=\frac{k(k-1)+(n-k)(n-k-1)}{2}$ $=k^2-nk+\frac{n^2-n}{2}$ This quadratic function of f is minimized at k = n/2 and maximized at k = 1 or k = n – 1 Hence, if G is not connected, then the number of edges does not exceed the value of this function at 1 and at n-1, namely $\frac{(n-1)(n-2)}{2}$.
3(b)	**If a graph G has exactly two vertices of odd degree, then prove that there is a path joining these two vertices.** **Proof:** Case (i): Let G be connected. Let v_1 and v_2 be the only vertices of G with are of odd degree. But we know that number of odd vertices is even. So clearly there is a path connecting v_1 and v_2, because G is connected. Case (ii): Let G be disconnected Then the components of G are connected. Hence v_1 and v_2 should belong to the same component of G. Hence, there is a path between v_1 and v_2.
4(a)	**Prove that a simple graph with n vertices and k components can have at most** $\frac{(n-k)(n-k+1)}{2}$ **edges.** Let the number of vertices of the ith component of G be , ≥ 1.. $\sum_{i=1}^{k} n_i = n \Rightarrow \sum_{i=1}^{k}(n_i-1)=(n-k)$ Then $\Rightarrow \left(\sum_{i=1}^{k}(n_i-1)\right)^2 = n^2-2nk+k^2$ that is $\sum_{i=1}^{k}(n_i-1)^2 \le n^2-2nk+k^2 \Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2-2nk+k^2+2n-k$ Now the maximum number of edges in the ith component of G = $\frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$ $\le \frac{(n^2-2nk+k^2+2n-k)}{2}-\frac{n}{2} \le \frac{(n-k)(n-k+1)}{2}$
4(b)	**If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.** **Solution:** Let *2n* be the number of vertices of the given graph….(1) Let n_e be the number of edges of the given graph. By Handshaking theorem, we have $\sum_{i=1}^{2n} \deg v_i = 2n_e$ $2nk = 2n_e$ (1) $n_e = nk$ Number of edges =multiple of . Hence the number of edges of the graph is a multiple of k

<table>
<tr><td>5(a)</td><td>Draw the graph with 3 vertices A,B,C, D & E such that the deg(A)=3,B is an odd vertex, deg(C)=2 and D and E are adjacent.
Solution:
d(E)=5 ,d(C)=2,d(D)=5 ,d(A)=3 d(B)=1

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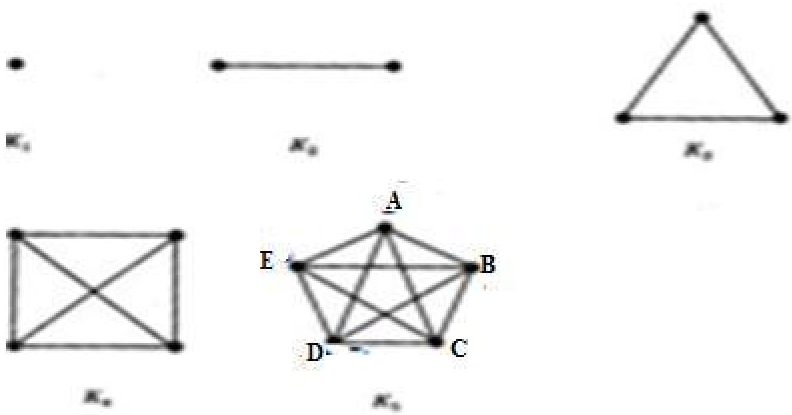
<tr><td>5(b)</td><td>Draw the complete graph K_5 with vertices A,B,C,D,E. Draw all complete sub graph of K_5 with 4 vertices.
Solution:


complete sub graph with 4 vertices
</td></tr>
<tr><td>6(a)</td><td>Prove that a given connected graph G is Euler graph if and only if all vertices of G are of even degree.
Solution:
Case (i) Prove If G is Euler graph→ Every vertex of G has even degree.
Case (ii) Prove If Every vertex of G has even degree.→ G is Euler graph (by Contradiction Method).</td></tr>
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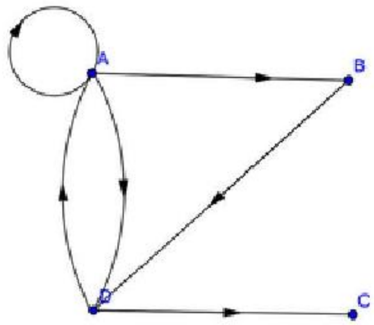
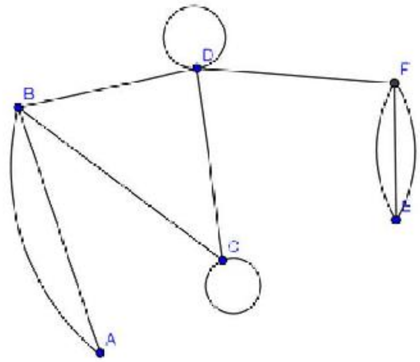
6(b) **Find the adjacency matrix of the given directed graph.**

(i)



(ii)



Answer:

$$(i)\begin{pmatrix} 1 & 10 & 1 \\ 0 & 001 & \\ 0 & 000 & \\ 1 & 0 & 10 \end{pmatrix} \qquad (ii)\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ & 201 & 100 & & & \\ 012 & & & 100 & & \\ & 011 & 200 & & & \\ & 000 & 003 & & & \\ 0 & 0 & 0 & 1 & 3 & 0 \end{pmatrix}$$

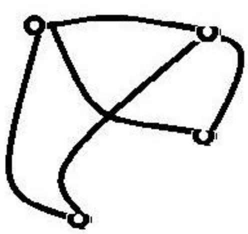
7(a) **Show that isomorphism of simple graphs is an equivalence relation. [November 2014]**

Solution:

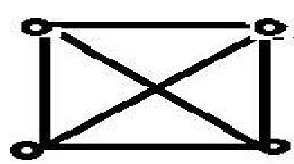
G is isomorphism to itself by the identity function, So isomorphism is reflexive. Suppose that G is isomorphic to H.Then there exists a one –to-one correspondence f from G to H that preserves adjacency and nonadjacency.It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and non-adjacency.Hence isomorphism is symmetric.If G is isomorphic to H and H is isomorphic to K then there are ono-to-one correspondences f and g from G to H and from H to K that preserves adjacency and nonadjacency.It follows that $g - f$ is a one-to-one correspondencies from G to K that preserves adjacency and non-adjacency.Hence isomorphism is transitive.

7(b) **Find the incidence matrix for the following graph.**

(i)



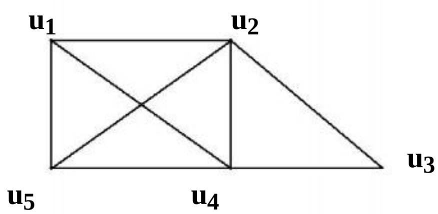
(ii)

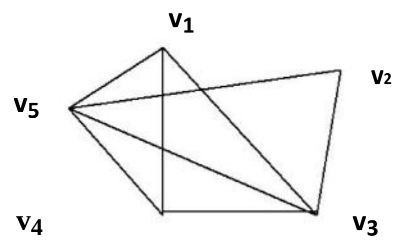


Answer:

$$(i)\begin{pmatrix}1 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & 0\end{pmatrix} \qquad (ii)\begin{pmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{pmatrix}$$

8(a) **Examine whether the following pair of graphs are isomorphic. If not isomorphic, give the reasons**





Solution:

Same number of vertices and edges. Also an equal number of vertices with a given degree.
The adjacency matrices of the two graphs are

$$\begin{pmatrix}0 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 0 & 1 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 0 & 1 & 0\end{pmatrix}$$

since the two adjacency matrices are the same, the two graphs are isomorphic.

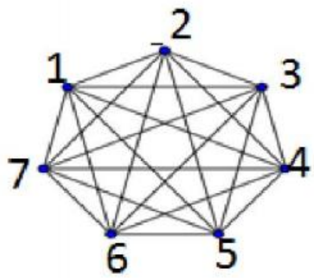
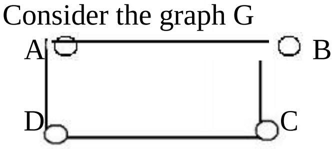
8(b) **Prove that if a graph G has not more than two vertices of odd degree, then there can be Euler path in G.**

Statement: Let the odd degree vertices be labeled as V and W in any arbitrary order. Add an edge to G between the vertex pair (V,W) to form a new graph G" .

Now every vertex of G" is of even degree and hence G" has an Eulerian Trail T.

If the edge that we added to G is now removed from T, It will split into an open trail containing all edges of G which is nothing but an Euler path in G

9(a) **Show that K_7 has Hamiltonian graph. How many edge disjoint Hamiltonian cycles are there in K_7 ? List all the edge-disjoint Hamiltonian cycles. Is it Eulerian graph ?**

	Solution: The Graph of K_7  K_7 has two edges disjoint Hamiltonian cycles. The edge disjoint Hamiltonian cycles are 1_2_3_4_5_6_7_1 and 1_3_6_2_4_7_5_1 K_7 is an Eulerian graph
9(b)	**Let G be a simple indirected graph with n vertices. Let u and v be two non adjacent vertices in G such that deg(u) + deg(v) ≥ n in G. Show that G is Hamiltonian if and only if G + uv is Hamiltonian.** **Solution:** If G is Hamiltonian, then obviously G + uv is also Hamiltonian. Conversely, suppose that G + uv is Hamiltonian, but G is not. Then by Dirac theorem, we have deg(u) + deg(v) < n which is a contradiction to our assumption. Thus G + uv is Hamiltonian implies G is Hamiltonian.
10(a)	**Draw a graph that is both Eulerian and Hamiltonian.** **Solution:** Example of Eulerian and Hamiltonian. Consider the graph G  n G, consider the cycle A-B-C-D-A. Since the cycle contains all the edges, G is Eulerian. Moreover, since the cycle contains all the vertices exactly once, G is Hamiltonian.
10(b)	**Prove that any 2 simple connected graphs with *n* vertices all of degree 2 are isomorphic.** **Proof:** We know that total degree of a graph is given by $\sum_{i=1}^{n} d(V_i) = 2\|E\|$ Then |V| = number of vertices n |E| = number of edges Further the degree of every vertex is 2. Therefore we have, $\sum_{i=1}^{n} 2 = 2\|E\|$ $2((n) - 1 + 1) = 2\|E\|$ $\Rightarrow n = \|E\|$ Hence number of edges = number of vertices. Hence they are isomorphic.