	UNIT II COMBINATORICS
	PART – A
1.	**State pigeon hole principle.** **Ans:** If (n+1) pigeons occupies n holes then at least one hole has more than 1 pigeon.
2.	**State the generalized pigeon hole principle.** **Ans:** If m pigeons occupies n holes ($m>n$), then at least one hole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.
3.	**Show that, among 100 people, at least 9 of them were born in the same month.** **Ans:** Here no.of pigeon =m= no. of people =100 No. of holes = n= no. of month =12 Then by generalized pigeon hole principle, $\left\lfloor \frac{100-1}{12} \right\rfloor + 1 = 9$ were born in the same month.
4.	**In how many ways can 6 persons occupy 3 vacant seats?** **Ans:** Total no of ways = $6c_3$ = 20 ways.
5.	**How many permutations of the letters in ABCDEFGH contain the string ABC .** **Ans:** Because the letters ABC must occur as block, we can find the answer by finding no of permutation of six objects, namely the block ABC and individual letters D,E,F,G and H . Therefore, there are 6! =720 permutations of the letters in ABCDEFGH which contains the string ABC.
6.	**How many different bit strings are there of length 7?** **Ans:** By product rule, 2^7=128 ways
7.	**How many ways are there to form a committee, if the committee consists of 3 educationalists and 4 socialist, if there are 9 educationalists and 11 socialist?** **Ans:** The 3 educationalist can be chosen from 9 educationalists in $9c_3$ ways. The 4 socialist can be chosen from 11 socialist in $11C_4$ ways. By product rule, the no of ways to select, the committee is = $9C_3.11C_4$ = 27720 ways.
8.	**There are 5 questions in a question paper in how many ways can a boy solve one or more questions?** **Ans:** The boy can dispose of each question in two ways .He may either solve it or leave it. Thus the no. of ways of disposing all the questions= 2^5 . But this includes the case in which he has left all the questions unsolved. The total no of ways of solving the paper = $2^5 - 1$ = 31.
9.	**If the sequence $a_n = 3.2^n$, $n \geq 1$, then find the corresponding recurrence relation.** **Ans:** For n ≥ 1 $a_n = 3.2^n$, $a_{n-1} = 3.2^{n-1} = 3.\frac{2^n}{2} \Rightarrow a_{n-1} = \frac{a_n}{2} \Rightarrow 2a_{n-1} = a_n$ $a_n = 2a_{n-1}$, for n≥1, with $a_0 = 3$.
10.	**If seven colours are used to paint 50 bicycles, then show that at least 8 bicycles will be the same colour.** **Ans:** Here, No. of Pigeon = = No. of bicycle=50 No. of Holes == No. of colours = 7 By generalized pigeon hole principle, we have $\left\lfloor \frac{50-1}{7} \right\rfloor + 1 = 8$

11.	**Find the recurrence relation whose solution is** $S(k) = 5.2^k$ **Ans:** Given $S(k) = 5.2^k \Rightarrow S(k-1) = 5.2^{k-1} = \frac{5}{2}.2^k \Rightarrow 2S(k-1) = 5.2^k = S(k)$ $2S(k-1) - S(k) = 0$, with $S(0) = 5$ is the required recurrence relation.
12.	**Find the associated homogeneous solution for** $a_n = 3a_{n-1} + 2n$. **Ans:** Its associated homogeneous equation is $a_n - 3a_{n-1} = 0$ Its characteristic equation is r-3 =0 $\Rightarrow$ r =3 Now, the solution of associated homogeneous equation is $a_n = A.3^n$
13.	**Solve** $S(k) - 7S(k-1) + 10S(k-2) = 0$ **Ans:** The associated homogeneous relation is $S(k) - 7S(k-1) + 10S(k-2) = 0$ Its characteristic equation is $r^2 - 7r + 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow$ r =2,5 The solution of associated homogeneous equation is $S_k = A.2^k + B.5^k$
14.	**Define Generating function.** **Ans:** The generating function for the sequence„s" with terms $a_0, a_1, ...a_n$ …,of real numbers is the infinite sum . G(*x*) = G(*s*,*x*) = $a_0 + a_1x + ... + a_nx^n + ... = \sum_{n=0}^{\infty} a_nx^n$.
15.	**Find the generating function for the sequence „s" with terms 1,2,3,4……..** **Ans:** $G(x) = G(s,x) = \sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + ... = (1-x)^{-2} = \frac{1}{(1-x)^2}$.
16.	**How many permutations of (a, b, c, d, e, f, g) end with a? [November 2014]** **Ans:** 6! $\times$ 1!=720
17.	**Find the number of arrangements of the letters in MAPPANASSRR.** **Ans:** Number of arrangements = $\frac{11!}{3!2!2!} = \frac{3991680}{48}$
18.	**In how many ways can letters of the word "INDIA" be arranged?** **Ans:** The word contains 5 letters of which 2 are I" s. The number of words possible = $\frac{5!}{2!} = 60$.
19.	**How many students must be in a class to guarantee that atleast two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points.** **Ans:** There are 101 possible scores as 0, 1, 2, …,100. By Pigeon hole principle, we have among 102 students there must be atleast two students with the same score. The class should contain minimum 102 students.
20	**Show that among any group of five (not necessarily consecutive) integers, there are two with same remainder when divided by 4.** **Ans:** Take any group of five integers. When these are divided by 4 each have some remainder. Since there are five integers and four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, atleast two have the same remainder.

PART – B

1(a)

Using Mathematical induction prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let P(n) : $1^2 + 2^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(1) Assume P(1) : $1^2 = \frac{1(1+1)(2.1+1)}{6}$ is true

(2) Assume P(k) : $1^2 + 2^2 + \ldots\ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true, where k is any integer.

(3) $P(k+1) = 1^2 + 2^2 + \ldots\ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{(k+1)[(k+1)+1][(2(k+1)+1]}{6}$$

Therefore P(k + 1) is true.

Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all n.

1(b)

Use mathematical Induction to prove that $(3^n + 7^n - 2)$ **is divisible by 8, for** $n \geq 1$.
Solution:

Let $P(n)$: $(3^n + 7^n - 2)$ is divisible by 8.

(i) $P(1)$: $(3^1 + 7^1 - 2)$ 8 is divisible by 8, is true.

(ii) Assume $P(k)$: $(3^k + 7^k - 2)$ is divisible by 8 is true ------(1)

Claim: $P(k+1)$ is true

$P(k+1) = 3^{k+1} + 7^{k+1} - 2$
$= 3 \cdot 3^k + 7 \cdot 7^k - 2$
$= 3.3^k + 3 \cdot 7^k + 4 \cdot 7^k - 6 + 4$
$= 3(3^k + 7^k$
$- 2) + 4(7^k + 1)$

$\therefore$ $4(7^k + 1)$ is divisible by 8 and by (1) $3(3^k + 7^k - 2)$ is divisible by 8. $P(k+1) = 3(3^k + 7^k - 2) + 4(7^k + 1)$ is divisible by 8 is true.

2(a)

Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ **is divisible by 43 for each positive integer** n.
Solution:

S(1): Inductive step: for = 1,

$6^{1+2} + 7^{2+1}$ =559, which is divisible by 43
So S(1) is true.
Assume S(k) is true (i.e) $6^{k+2} + 7^{2k+1} = 43m$ for some integer m.

	To prove S(k+1) is true.Now $6^{k+3}+7^{2k+3}=6^{k+3}+7^{2k+1}.7^{2}$ $=6(6^{k+2}+7^{2k+1})+43.7^{2k+1}$ $=6.43m+43.7^{2k+1}$ $=43(6m+7^{2k+1})$ Which is divisible by 43. So S(k+1) is true. By Mathematical Induction , S(*n*) is true for all integer n.
2(b)	**Using mathematical induction ,prove that** $2+2^{2}+2^{3}+.........+2^{n}=2^{n+1}-2$ Let p (n) = $2+2^{2}+2^{3}+.........+2^{n}$. Assume p (1): $2^{1}=2^{1+1}-2$ is true. Assume p(k) : $2+2^{2}+2^{3}+.........+2^{k}=2^{k+1}-2$ is true Claim p(k+1) is true. P(k+1) : $2+2^{2}+2^{3}+...+2^{k}+2^{k+1}=2^{k+1}-2+2^{k+1}=2.2^{k+1}-2=2^{k+2}-2$ P(k+1) is true. Hence it is true for all n.
3(a)	**Suppose there are six boys and five girls,** **(1) In how many ways can they sit in a row.** **(2) In how many ways can they sit in a row, if the boys and girls each sit together.** **(3) In how many ways can they sit in a row, if the girls are to sit together and the boy don□ t sit together.** **(4) How many seating arrangements are there with no two girls sitting together.** **Solution:** 1. There are 6 + 5 = 11 persons and they can sit in $11P_{11}$ ways. $11P_{11}$ = 11! ways 2. The boys among themselves can sit in 6! ways and girls among themselves can sit in 5! ways.They can be considered as 2 units and can be permuted in 2! ways. Thus the required seating arrangement can be done in = 2! x 6! x 5! ways = 172800 ways 3. The boys can sit in 6! Ways and girls in 5! ways. Since girls have to sit together they are considered as one unit. Among the 6 boys either 0 or 1 or 2 or 3 or 4 or 5 or 6 have to sit to the left of the girls units. Of these seven ways 0 and 6 cases have to be omitted as the boys do not sit together.Thus the required number of arrangements = 5 x 6! x 5! = 432000 ways. 4. The boys can sit in 6 ! ways. There are seven places where the girls can be placed. Thus the total arrangements are $7P_{5}$ x 6! Ways = 1814400 ways.
3(b)	**A bit is either 0 or 1 . A byte is a sequence of 8 bits. Find the number of bytes.Among these how many are (i) Starting with 11 and ending with 00 (ii) Starting with 11 but not ending with 00.** **Solution:** (1) Consider a byte starting with 11 and ending with 00.Now the remaining 4 places can be filled with either 0 or 1 which can be done in 2^{4}.Hence there are 16 bytes starting with 00 and ending with11. (2) Consider a byte starting with 11 and not ended with 00 Now there are 3 bytes which is not ended with 00(ended with 01,10 and 11).Now the remaining 4 places can be filled with either 0 or 1 which can be done in 2^{4}ways.Hence there are 3×16=48 bytes starting with 00 but not ending with11
4(a)	**How many positive integers '*n* ' can be formed using the digits 3,4,4,5,5,6,7 if '*n* ' has to exceed 50,00,000 ?** **Solution:** Consider a 7digit number $p_1, p_2, p_3, p_4, p_5, p_6, p_7$, in order to be a number ≥ 5000000 , p_1 is filled with

either 5 or 6 or 7 (mutually exclusive)

Case(1): p_1 is filled with 5 and remaining 6 position are filled with 3, 4, 4(repeated),5,6,7 in $= \frac{6!}{0\ 2!} = 36$

Case(2): p_1 is filled with 6 and remaining 6 positions are filled with 3,4,4 (repeated) 5,5 (repeated), 7 in

$= \frac{6!}{2!\,2!} = 180$

Case(3) p_1 is filled with 7 and remaining 6 position are filled with 3,4,4(repeated),5,5 (repeated), 6 in

$= \frac{6!}{2!\,2!} = 180$

All above 3 cases are mutually exclusive in total 360+180+180=720 ways.

4(b) **Prove that in any group of six people there must be atleast three mutual friends or three mutual enemies.**

Proof:

Let the six people be A, B, C, D, E and F. Fix A. The remaining five people can accommodate into two groups namely

(1) Friends of A and (2) Enemies of A

Now by generalized Pigeon hole principle, at least one of the group must contain $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ people.

Let the friend of A contain 3 people.(Let it be B, C, D)

Case(1) If any two of these three people (B, C, D) are friends, then these two together with A form three mutual friends.

Case(2) If no two of these three people are friends, then these three people (B, C, D) are mutual enemies. In either case, we get the required conclusion.

If the group of enemies of A contains three people, by the above similar argument, we get the required conclusion.

5(a) **A computer password consists of a letter of English alphabet followed by 2 or 3 digits. Find the following :**

(1) The total number of passwords that can be formed

(2) The number of passwords that no digit repeats.

Sol: (1) Since there are 26 alphabets and 10 digits and the digits can be repeated by the product rule the number of 3-character password is 26.10.10=2600

Similarly the number of 4 character password is 26.10.10.10=26000

Hence the tool number of password is 2600+26000=28600.

(2) Since the digits are not repeated, the first digit after alphabet can be taken from any one out of 10, the second digit from remaining 9 digits and so on.

Thus the number of 3-character password is 26.10.9=2340

Similarly the number of 4- character password is 26.10.9.8=18720

Hence the total number of password is 2340+18720=21060.

5(b) **Show that among (+) positive integers not exceeding 2*n* there must be an integer that divides one of the other integers.**

Solution:

Let the (+ 1) integers be $a_1, a_2, \ldots a_{n+1}$

Each of these numbers can be expressed as an odd multiple of a power of 2.

i.e $a_i = 2^{k_i} \times m_i$

Where k_i non negative integer

m_i odd integer where i =1, 2,3,..., n +1.

Here, Pigeon=The odd positive integers $m_1, m_2, \ldots m_{n+1}$ less than $2n$

Pigeon= 'n' odd positive integer less than $2n$.

Hence by pigeon hole principle, 2 of the integers must be equal.

Now $a_i = 2^{k_i} m_i$ and $a_j = 2^{k_j} m_j$

$$\frac{a_i}{a_j} = \frac{2^{k_i}}{2^{k_j}} \quad (m_i = m_j)$$

Case-1: If $k_i < k_j$ then 2^{k_i} divides 2^{k_j} and hence a_i divides a_j.

Case-2: If $k_i > k_j$ then a_j divides a_i.

6(a) **In A survey of 100 students, it was found that 30 studied Mathematics, 54 studied Statistics, 25 studied Operations Research, 1 studied all the three subjects, 20 studied Mathematics and Statistics, 3 studied Mathematics and Operation Research and 15 studied Statistics and Operation Research. Find how many students studied none of these subjects and how many students studied only Mathematics?**

Solution.

n(A) = 30; n(B) = 54;n(C) = 25;

n(A∩B) = 20; n(A∩C) = 3; n(B∩C) = 15;

n(A∩B∩C)=1

n(A∪B∪C) = n(A)+n(B)+n(C) – n(A∩B) – n(B∩C) – n(A∩C) + n(A∩B∩C) = 72

None of the subjects = 28.

Only mathematics = 8.

6(b) **A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and Russian, 23 have taken courses in both Spanish and French and 14 have taken courses in both French and Russian. If 2092 students have taken atleast one of Spanish, French and Russian, how many students have taken a course in all three languages?**

Solution: S-Spanish,F-French, R-Russian

|S|=1232 |F|=879 |R|=114 |S∩R|=103 |S∩F|=23 |F∩R|=14

|S∪F∪R|=2092

|S∪F∪R=|S|+|F|+|R|-|S∩F|- |S∩R|-|F∩R|+|S∩F∩R|

∴ |S∩F∩R|=7

7(a) **Find all the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$**

Solution:

Given non-homogeneous equation can be written as $a_n - 5a_{n-1} + 6a_{n-2} - 7^n = 0$

Now, its associated homogeneous equation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Its characteristic equation is $r^2 - 5r + 6 = 0$

Roots are r = 2,3

Solution is $a_n^{(h)} = c_1 2^n + c_2 3^n$

To find particular solution

Since $F(n) = 7^n$, then the solution is of the form $C.7^n$, where C is a constant.

Therefore, the equation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ becomes $C7^n = 5C7^{n-1} - 6C7^{n-2} + 7^n$ ……(1)

Dividing the both sides of (1) by 7^{n-2}.

$$(1) \rightarrow \frac{C.7^n}{7^{n-2}} = \frac{5C7^{n-1}}{7^{n-2}} - \frac{6C7^{n-2}}{7^{n-2}} + \frac{7^n}{7^{n-2}} \rightarrow C = \frac{49}{20}$$

	Hence the particular solution is $a_n^{(p)} = \left(\frac{49}{20}\right) 7^n$ Therefore, $a_n = c_1(2)^n + c_2(3)^n + \left(\frac{49}{20}\right) 7^n$
7(b)	**Find the number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5 &7.** Sol: Let A, B, C,D are the set of integers between 1 and 250 that are divisible by 2, 3, 5, 7 respectively. $\therefore \; \lvert A\rvert = \left[\frac{250}{2}\right] = 125, \quad \lvert B\rvert = \left[\frac{250}{3}\right] = 83$ $\lvert C\rvert = \left[\frac{250}{5}\right] = 50, \quad \lvert D\rvert = \left[\frac{250}{7}\right] = 35$ $\lvert A\cap B\rvert = \left\lfloor \frac{250}{LCM(2,3)} \right\rfloor = \left\lfloor \frac{250}{2\times 3} \right\rfloor = \left\lfloor \frac{250}{6} \right\rfloor = 41$ $\lvert A\cap C\rvert = \left\lfloor \frac{250}{LCM(2,5)} \right\rfloor = \left\lfloor \frac{250}{2\times 5} \right\rfloor = \left\lfloor \frac{250}{10} \right\rfloor = 25$ $\lvert A\cap D\rvert = \left\lfloor \frac{250}{LCM(2,7)} \right\rfloor = \left\lfloor \frac{250}{2\times 7} \right\rfloor = \left\lfloor \frac{250}{14} \right\rfloor = 17$ $\lvert B\cap C\rvert = \left\lfloor \frac{250}{LCM(3,5)} \right\rfloor = \left\lfloor \frac{250}{5\times 3} \right\rfloor = \left\lfloor \frac{250}{15} \right\rfloor = 16$ $\lvert B\cap D\rvert = \left\lfloor \frac{250}{LCM(7,3)} \right\rfloor = \left\lfloor \frac{250}{7\times 3} \right\rfloor = \left\lfloor \frac{250}{21} \right\rfloor = 11$ $\lvert C\cap D\rvert = \left\lfloor \frac{250}{LCM(5,7)} \right\rfloor = \left\lfloor \frac{250}{5\times 7} \right\rfloor = \left\lfloor \frac{250}{35} \right\rfloor = 7$ $\lvert A\cap B\cap C\rvert = \left\lfloor \frac{250}{LCM(2,3,5)} \right\rfloor = \left\lfloor \frac{250}{2\times 3\times 5} \right\rfloor = 8$ $\lvert A\cap B\cap D\rvert = \left\lfloor \frac{250}{LCM(2,3,7)} \right\rfloor = \left\lfloor \frac{250}{2\times 3\times 7} \right\rfloor = 5$ $\lvert A\cap C\cap D\rvert = \left\lfloor \frac{250}{LCM(2,5,7)} \right\rfloor = \left\lfloor \frac{250}{2\times 5\times 7} \right\rfloor = 3$ $\lvert B\cap C\cap D\rvert = \left\lfloor \frac{250}{LCM(3,5,7)} \right\rfloor = \left\lfloor \frac{250}{3\times 5\times 7} \right\rfloor = 2$ $\lvert A\cap B\cap C\cap D\rvert = \left\lfloor \frac{250}{LCM(2,3,5,7)} \right\rfloor = \left\lfloor \frac{250}{2\times 3\times 5\times 7} \right\rfloor = 1$ $\lvert A\cup B\cup C\cup D\rvert = \lvert A\rvert+\lvert B\rvert+\lvert C\rvert+\lvert D\rvert-\lvert A\cap B\rvert-\lvert A\cap C\rvert-\lvert A\cap D\rvert-\lvert B\cap C\rvert$ $-\lvert B\cap D\rvert+\lvert C\cap D\rvert+\lvert A\cap B\cap C\rvert+\lvert A\cap B\cap D\rvert+\lvert A\cap C\cap D\rvert$ $+\lvert B\cap C\cap D\rvert-\lvert A\cap B\cap C\cap D\rvert$ =125+83+50+35-41-25-17-16-11-7+8+5+3+2-1=193 The number of integers between 1 and 250 that is divisible by any of the integers 2, 3, 5 and 7=193 Therefore not divisible by any of the integers 2, 3, 5 and 7=250-193=57.

8(a) **Solve the recurrence relation** $a_n = 2(a_{n-1} - a_{n-2})$ **where** $\geq$ and $a_0 = 1, a_1 = 2$

$a_n = 2(a_{n-1} - a_{n-2})$

$= a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is given by

$ª^2 - 2ª + 2 = 0$

$\therefore ª = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$

$\therefore ª = 1 + i, 1 - i$

$\therefore$ Solution is $a_n = A(1+i)^n + B(1-i)^n$

Where A and B are arbitrary constants

Now, we have

$z = x + iy$

$= r[\cos - + i \sin -]$

$- = \tan^{-1}\left(\frac{y}{x}\right)$

By Demoivre" s theorem we have,

$(1+i)^n = [\sqrt{2}\left(\cos\frac{®}{4} + i\sin\frac{®}{4}\right)]^n$

$= [\sqrt{2}]^n\left(\cos\frac{n®}{4} + i\sin\frac{n®}{4}\right)$

and $(1-i)^n = [\sqrt{2}]^n\left(\cos\frac{n®}{4} - i\sin\frac{n®}{4}\right)$

Now,

$a_n = A[[\sqrt{2}]^n\left(\cos\frac{n®}{4} + i\sin\frac{n®}{4}\right) + B[[\sqrt{2}]^n\left(\cos\frac{n®}{4} - i\sin\frac{n®}{4}\right)]$

$= [\sqrt{2}]^n\left((A+B)\cos\frac{n®}{4} + i(A-B)\sin\frac{n®}{4}\right)$

$\therefore a_n = [\sqrt{2}]^n\left(C_1\cos\frac{n®}{4} + C_2\sin\frac{n®}{4}\right)]$ (1)

Is the required solution. Let $C_1 = +$, $C_2 = (-)$

Since 0 = 1, 1 = 2

$(1) \Rightarrow a_0 = (\sqrt{2})[C_1 \cos 0 + C_2 \sin 0] = 0$

$\Rightarrow 1 = C_1$

$a_1 = [\sqrt{2}]^1 \left(C_1 \cos \frac{\pi}{4} + C_2 \sin \frac{\pi}{4} \right)]$

$2 = \sqrt{2}\left(C_1 \frac{1}{\sqrt{2}} + C_2 \sin \frac{1}{\sqrt{2}} \right)]$

$\Rightarrow 2 = C_1 + C_2$

$\Rightarrow C_2 = 1$

$\therefore a_n = [\sqrt{2}]^n \left(\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right)]$

8(b)

Solve the recurrence relation of the Fibonacci sequence of numbers $f_n = f_{n-1} + f_{n-2}, n > 2$ **with initial conditions** $f_1 = 1, f_2 = 1$.

Sol: The sequence of Fibonacci numbers satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ ..... (1) and satisfies the initial conditions $f_1 = 1, f_2 = 1$.

$(1) \Rightarrow f_n - f_{n-1} - f_{n-2} = 0$ ...(2)

Let be a solution of the given equation.

The characteristic equation is $^2 - - 1 = 0$

$r = \frac{1 \pm \sqrt{1+4}}{2}$

Let $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$

$\therefore$ By theorem

$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$ ...(3)

$f_1 = 1 \Rightarrow f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$

$(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2$ ...(4)

$f_2 = 1 \Rightarrow f_2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^2 + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^2 = 1$

$= \alpha_1 \frac{(1+\sqrt{5})^2}{4} + \alpha_2 \frac{(1-\sqrt{5})^2}{4} = 1$

$= (1+\sqrt{5})^2 \alpha_1 + (1-\sqrt{5})^2 \alpha_2 = 4$ ...(5)

	$(4)\times(1-\sqrt{5})\Rightarrow$ $(1-\sqrt{5})(1+\sqrt{5})\sim_1+(1-\sqrt{5})^2\sim_2=2(1-\sqrt{5})\ldots(6)$ $(6)-(5)\Rightarrow\sim_1(1+\sqrt{5})[1-\sqrt{5}-1-\sqrt{5}]=2-2\sqrt{5}-4$ $\sim_1(1+\sqrt{5})[-2\sqrt{5}]=-2-2\sqrt{5}$ $\sim_1(1+\sqrt{5})[-2\sqrt{5}]=-2(1+\sqrt{5})$ $\sim_1=\frac{1}{\sqrt{5}}$ $4)\Rightarrow(1+\sqrt{5})\frac{1}{\sqrt{5}}+(1-\sqrt{5})\sim_2=2$ $\frac{1}{\sqrt{5}}+1+(1-\sqrt{5})\sim_2=2$ $(1-\sqrt{5})\sim_2=2-\frac{1}{\sqrt{5}}-1$ $=1-\frac{1}{\sqrt{5}}$ $(1-\sqrt{5})\sim_2=\frac{\sqrt{5}-1}{\sqrt{5}}$ $\sim_2=\frac{-1}{\sqrt{5}}$ $(3)\Rightarrow f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$
9(a)	**Solve the recurrence relation** $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$ **[November 2014]** **Solution:** The unique Solution to this recurrence relation and the given initial condition is the sequence $\{a_n\}$ with $a_n = 1 - 2^n + 2 . 3^n$
9(b)	**A factory makes custom sports cars at an interesting rate. In the first month only one car is made, in the second month two cars are made and so on, with n cars made in the nth month.** **(1) Set up recurrence relation for the number of cars produced in the first *n* months by this factory.** **(2) How many cars are produced in the first year?** **Solution:** (i) $a_n = n + a_{n-1}$, $a_o = 0$ ($a_1 = 1, a_2 = 2 + a_1, etc$) (ii) By recursively $a_{12} = 78$
10(a)	**Solve** (+) − ( ) = , **with** ( ) = **and** ≥ **Solution**: Given $_{+1} - 2 - 4 = 0$ Multiply by x^n, and sum over all $n = 0$ to ∞.

$$\sum_{n=0}^{\infty} a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 4^n x^n = 0$$

$$G(x) = \frac{1-3x}{(1-2x)(1-4x)}$$

By Applying Partial fractions we get $A = \frac{1}{2}, B = \frac{1}{2}$

$$G(x) = \frac{1}{2}\sum_{n=0}^{\infty} 2^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 4^n x^n$$

hence we get

$$a_n = 2^{n-1} + 2(4)^{n-1}$$

10(b) **Find the generating function of Fibonacci sequence.**

Solution

Fibonacci sequence : $f_n = f_{n-1} + f_{n-2}, \ n \geq 2$ with $f_o = 0, \ f_1 = 1$

Multiply by z^n , and sum over all $n \geq 2$.

$$\sum_{n=2}^{\infty} f_n z^n = \sum_{n=2}^{\infty} f_{n-1} z^n + \sum_{n=2}^{\infty} f_{n-2} z^n$$

$$G(z) - f_0 - f_1 z = z(G(z) - f_0) + z^2(G(z))$$

$$G(z) = \sum_{n=0}^{\infty} f_n z^n$$

Where $(i.e)\ G(z) - zG(z) - z^2 G(z) = f_0 + f_1 z - z f_0$

$$G(z) = \frac{z}{1 - z - z^2}$$