UNIT IV – FOURIER TRANSFORM

PART –A

1. **Define Fourier Integral theorem.**

ANS

If f(x) is piece wise continuously differentiable and absolutely integrable in (-∞,∞), then

$$f(x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t).e^{is(x-t)}dt.ds$$

2. **Define Fourier Transform pair and its Parseval's identity.**

ANS

- The Fourier Transform of f(x) is $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx=F[s]$
- The inverse Fourier Transform is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[s].e^{-isx}ds$
- Parseval's identity $\int_{-\infty}^{\infty} F[s]^2 ds=\int_{-\infty}^{\infty} f(x)^2 dx$

3. **Find the Fourier Transform of** $f(x)=\begin{cases} e^{ikx}, a<x<b \\ 0, x<a \text{ and } x>b \end{cases}$

ANS

The Fourier transform of f(x) is $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx}.e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{i(k+s)x}dx$$

$$F[s]=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)x}}{i(k+s)}\right]_{x=a}^{b}=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(k+s)b}}{i(k+s)}-\frac{e^{i(k+s)a}}{i(k+s)}\right]$$

4. **State Convolution theorem on Fourier Transform**

ANS

F[f(x)*g(x)]=F[f(x)].F[g(x)] where $f(x)*g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t).g(x-t).dt$

5. **State Fourier cosine transform pair and state parseval's identity on it.**

ANS

- The Fourier cosine transform of f(x) is $F_C[f(x)]=\frac{2}{\sqrt{2\pi}}\int_0^{\infty} f(x).\cos(sx).dx=F_C(s)$

- The Inverse Fourier cosine Transform is $f(x)=\frac{2}{\sqrt{2\pi}}\int_0^\infty F_C(s).\cos(sx).ds$
- Parseval's identity: $\int_0^\infty F_C(s)^2 ds=\int_0^\infty f(x)^2 dx$

6. **State Fourier sine transform pair and state parseval's identity on it.**

ANS

- The Fourier sine transform of f(x) is $F_S[f(x)]=\frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\sin(sx).dx=F_S(s)$
- The Inverse Fourier sine Transform is $f(x)=\frac{2}{\sqrt{2\pi}}\int_0^\infty F_S(s).\sin(sx).ds$
- Parseval's identity: $\int_0^\infty F_S(s)^2 ds=\int_0^\infty f(x)^2 dx$

7. **Find the Fourier cosine and sine transform of e^{-ax},a>0.**

ANS

$$\text{The Fourier Cosine Transform is } F_C[f(x)]=\frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\cos(sx).dx=\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-ax}.\cos(sx).dx=\frac{2}{\sqrt{2\pi}}\left[\frac{a}{a^2+s^2}\right]$$

$$\text{The Fourier Sine Transform is } F_S[f(x)]=\frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\sin(sx).dx=\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-ax}.\sin(sx).dx=\frac{2}{\sqrt{2\pi}}\left[\frac{s}{a^2+s^2}\right]$$

8. **If $F_c[s]$ is the Fourier cosine transform of f(x), prove that** $F_C[f(x)\cos(ax)]=\frac{1}{2}[F_C(s+a)+F_C(s-a)]$ **.(OR) State and prove Modulation property**

ANS

$$\text{The Fourier Cosine Transform is } F_c[f(x)] = \frac{2}{\sqrt{2\pi}}\int_0^\infty f(x)\cos(sx)dx = F_c[s].$$

$$F_c[f(x).\cos(ax)] = \frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\cos(ax).\cos(sx)dx$$

$$\text{WKT } \cos A.\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$

$$F_c[f(x).\cos(ax)] = \frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\left[\frac{\cos(a+s)x+\cos(a-s)x}{2}\right]dx$$

$$= \frac{1}{2}\left\{\frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).[\cos(a+s)x+\cos(a-s)x]dx\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\cos(s+a)x.dx + \frac{2}{\sqrt{2\pi}}\int_0^\infty f(x).\cos(s-a)x.dx\right\}$$

$$F_c[f(x).\cos(ax)] = \frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right]$$

9. Find the function f(x) whose sine transform is $\frac{e^{-as}}{s}$.

ANS

Given $F_s[f(x)] = \frac{e^{-as}}{s}$.

$$\text{The Inverse Fourier Sine transform is } f(x) = \frac{2}{\sqrt{2\pi}}\int_0^\infty F_s[s].\sin(sx).ds$$

$$f(x) = \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-as}}{s}.\sin(sx).ds$$

$$f'(x) = \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-as}}{s}.s.\cos(sx).ds = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-as}.\cos(sx).ds = \frac{2}{\sqrt{2\pi}}\frac{a}{a^2+x^2}$$

$$f'(x) = \frac{2}{\sqrt{2\pi}}\frac{a}{a^2+x^2} \Rightarrow f(x) = \int \frac{2}{\sqrt{2\pi}}\frac{a}{a^2+x^2}.dx = \frac{2a}{\sqrt{2\pi}}.\int\frac{dx}{a^2+x^2} = \frac{2a}{\sqrt{2\pi}}.\frac{1}{a}.\tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore f(x) = \frac{2}{\sqrt{2\pi}}.\tan^{-1}\left(\frac{x}{a}\right)$$

10. Find the function f(x) whose sine transform is e^{-as}.

ANS

The Inverse Fourier Sine transform is $f(x)=\frac{2}{\sqrt{2\pi}}\int_0^\infty F_S[s].\sin(sx).ds$

$$f(x)=\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-as}.\sin(sx).ds=\frac{2}{\sqrt{2\pi}}\left[\frac{x}{a^2+x^2}\right]$$

PART – B

1. Find the Fourier transform of $f(x)=\begin{cases}1, |x|<a\\ 0, |x|\geq a\end{cases}$. Hence deduce that (i) $\int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2}$. (ii) $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$.
2. Find the Fourier transform of $f(x)=\begin{cases}a-|x|, |x|<a\\ 0, |x|\geq a\end{cases}$. Hence deduce that (i) $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$. (ii) $\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$.
3. Find the Fourier transform of $f(x)=\begin{cases}a^2-x^2, |x|<a\\ 0, |x|\geq a\end{cases}$. Deduce that (i) $\int_0^\infty \frac{\sin(s)-s.\cos(s)}{s^3}.\cos\left(\frac{s}{2}\right)ds=\frac{3\pi}{16}$. (ii) $\int_0^\infty \frac{\sin(t)-t.\cos(t)}{t^3}dt=\frac{\pi}{4}$. (iii) $\int_0^\infty\left[\frac{\sin(t)-t.\cos(t)}{t^3}\right]^2 dt=\frac{\pi}{15}$.
4. Find the Fourier transform of f(x) = $e^{-a|x|}$. Deduce that (i) $\int_0^\infty \frac{\cos(sx)}{s^2+a^2}ds=\frac{\pi}{2a}f(x)$ (ii) $\int_0^\infty \frac{dx}{x^2+a^2}=\frac{\pi}{2a}$ (iii) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$
5. Definition of Self reciprocal.
6. Find the Fourier transform of f(x) = $e^{-a^2x^2}$. Show that $e^{-x^2/2}$ is self reciprocal under Fourier Transform.
7. Find the Fourier Sine transform of f(x) = e^{-ax}. Deduce the inversion formula on it. Deduce that $\int_0^\infty \frac{x^2.dx}{(x^2+a^2)^2}=\frac{\pi}{4a}$
8. Find the Fourier cosine transform of f(x) = $e^{-a^2x^2}$. Show that $e^{-x^2/2}$ is self reciprocal under Fourier Cosine Transform.
9. Evaluate (i) $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ *and* (ii) $\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.
10. Find the Fourier cosine and sine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.