

UNIT IV – FOURIER TRANSFORM

PART –A

1. Define Fourier Integral theorem.

ANS

If $f(x)$ is piece wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{is(x-t)} dt \cdot ds$$

2. Define Fourier Transform pair and its Parseval's identity.

ANS

◆ The Fourier Transform of $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx = F[s]$

◆ The inverse Fourier Transform is $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[s] \cdot e^{-isx} ds$

◆ Parseval's identity $\int_{-\infty}^{\infty} F[s]^2 ds = \int_{-\infty}^{\infty} f(x)^2 dx$

3. Find the Fourier Transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$

ANS

The Fourier transform of $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x} dx$$

$$F[s] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_{x=a}^b = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)b}}{i(k+s)} - \frac{e^{i(k+s)a}}{i(k+s)} \right]$$

4. State Convolution theorem on Fourier Transform

ANS

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)] \text{ where } f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \cdot dt$$

5. State Fourier cosine transform pair and state parseval's identity on it.

ANS

► The Fourier cosine transform of $f(x)$ is $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(sx) \cdot dx = F_c(s)$

▶ The Inverse Fourier cosine Transform is $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_c(s) \cdot \cos(sx) \cdot ds$

▶ Parseval's identity: $\int_0^{\infty} F_c(s)^2 ds = \int_0^{\infty} f(x)^2 dx$

6. State Fourier sine transform pair and state parseval's identity on it.

ANS

✗ The Fourier sine transform of $f(x)$ is $F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \sin(sx) \cdot dx = F_s(s)$

✗ The Inverse Fourier sine Transform is $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s(s) \cdot \sin(sx) \cdot ds$

✗ Parseval's identity: $\int_0^{\infty} F_s(s)^2 ds = \int_0^{\infty} f(x)^2 dx$

7. Find the Fourier cosine and sine transform of e^{-ax} , $a > 0$.

ANS

The Fourier Cosine Transform is $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot \cos(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \left[\frac{a}{a^2 + s^2} \right]$

The Fourier Sine Transform is $F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \sin(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot \sin(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \left[\frac{s}{a^2 + s^2} \right]$

8. If $F_c[s]$ is the Fourier cosine transform of $f(x)$, prove that $F_c[f(x) \cos(ax)] = \frac{1}{2} [F_c(s+a) + F_c(s-a)]$

.(OR) State and prove Modulation property

ANS

The Fourier Cosine Transform is $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos(sx) dx = F_c[s]$.

$$F_c[f(x) \cdot \cos(ax)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(ax) \cdot \cos(sx) dx$$

$$\text{WKT } \cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\begin{aligned} F_c[f(x) \cdot \cos(ax)] &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \left[\frac{\cos(a+s)x + \cos(a-s)x}{2} \right] dx \\ &= \frac{1}{2} \left\{ \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot [\cos(a+s)x + \cos(a-s)x] dx \right\} \\ &= \frac{1}{2} \left\{ \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(s+a)x \cdot dx + \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(s-a)x \cdot dx \right\} \end{aligned}$$

$$F_c[f(x) \cdot \cos(ax)] = \frac{1}{2} [F_c(s+a) + F_c(s-a)]$$

9. Find the function $f(x)$ whose sine transform is $\frac{e^{-as}}{s}$.

ANS

$$\text{Given } F_s[f(x)] = \frac{e^{-as}}{s}$$

The Inverse Fourier Sine transform is $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s[s] \cdot \sin(sx) \cdot ds$

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \cdot \sin(sx) \cdot ds$$

$$f'(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-as} \cdot s \cdot \cos(sx) \cdot ds = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-as} \cdot \cos(sx) \cdot ds = \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2}$$

$$f'(x) = \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2} \Rightarrow f(x) = \int \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2} \cdot dx = \frac{2a}{\sqrt{2\pi}} \cdot \int \frac{dx}{a^2 + x^2} = \frac{2a}{\sqrt{2\pi}} \cdot \frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore f(x) = \frac{2}{\sqrt{2\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

10. Find the function $f(x)$ whose sine transform is e^{-as} .

ANS

The Inverse Fourier Sine transform is $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s[s] \cdot \sin(\xi x) \cdot ds$

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-as} \cdot \sin(\xi x) \cdot ds = \frac{2}{\sqrt{2\pi}} \left[\frac{x}{a^2 + x^2} \right]$$

PART – B

- Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$. Hence deduce that (i) $\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$. (ii) $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.
- Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| \geq a \end{cases}$. Hence deduce that (i) $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (ii) $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$.
- Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| \geq a \end{cases}$. Deduce that
 (i) $\int_0^{\infty} \frac{\sin(s) - s \cdot \cos(s)}{s^3} \cdot \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$. (ii) $\int_0^{\infty} \frac{\sin(t) - t \cdot \cos(t)}{t^3} dt = \frac{\pi}{4}$. (iii) $\int_0^{\infty} \left[\frac{\sin(t) - t \cdot \cos(t)}{t^3}\right]^2 dt = \frac{\pi}{15}$.
- Find the Fourier transform of $f(x) = e^{-a|x|}$. Deduce that
 (i) $\int_0^{\infty} \frac{\cos(\xi x)}{s^2 + a^2} ds = \frac{\pi}{2a} f(x)$ (ii) $\int_0^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$ (iii) $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$
- Definition of Self reciprocal.
- Find the Fourier transform of $f(x) = e^{-a^2 x^2}$. Show that $e^{-x^2/2}$ is self reciprocal under Fourier Transform.
- Find the Fourier Sine transform of $f(x) = e^{-ax}$. Deduce the inversion formula on it. Deduce that
 $\int_0^{\infty} \frac{x^2 \cdot dx}{(x^2 + a^2)^2} = \frac{\pi}{4a}$
- Find the Fourier cosine transform of $f(x) = e^{-a^2 x^2}$. Show that $e^{-x^2/2}$ is self reciprocal under Fourier Cosine Transform.
- Evaluate (i) $\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$ and (ii) $\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}$ using transforms.
- Find the Fourier cosine and sine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.

UNIT-IV- FOURIER TRANSFORMS

PART - A

1. State Fourier integral theorem
2. Write the Fourier transform pair
3. State parseval's identity for fourier transform
4. State the convolution theorem for Fourier transform
5. Prove that $F[e^{ax}f(x)] = F(s+a)$ where $F[f(x)] = F(s)$.
6. If $F(s)$ is the Fourier transform of $f(x)$, show that $F[f(x-a)] = e^{isa}F(s)$. (OR) State and prove the change of scale property of Fourier Transform
7. If $F[f(x)] = F(s)$, then $F[f(x)\cos(ax)]$
8. Find the fourier transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$
9. Write down the Fourier cosine transform pair of formulae
10. Find the Fourier cosine transform of e^{-ax} .
11. If $F_c(s)$ is the Fourier cosine transform of $f(x)$, prove that $F_c[f(x)\cos(ax)] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$
12. If $F_c(s)$ is the Fourier cosine transform of $f(x)$, prove that the Fourier cosine transform of $f(ax)$ is $\frac{1}{a}F_c\left(\frac{s}{a}\right)$
13. Find the Fourier sine transform of e^{-ax} , $a > 0$.
14. Find the Fourier sine transform of $\frac{1}{x}$
15. Find the function $f(x)$ whose sine transform is e^{-as} .

PART - B

1. Find the Fourier integral representation of $f(x)$ defined as $f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$
2. Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$ and hence find the value of $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ and $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$
3. Derive the parseval's identity for Fourier Transforms
4. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ and hence evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$
5. Show that the Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

6. Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and using parseval's identity prove that

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

7. Find the Fourier transform of $e^{-|x|}$ and deduce that $\int_0^{\infty} \frac{\cos(\lambda x)}{x^2 + 1} dx = \frac{\pi}{2} e^{-|\lambda|}$

8. Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

9. Using Fourier cosine transform method, evaluate $\int_0^{\infty} \frac{dx}{(a^2 + x^2)(b^2 + x^2)}$ and hence Evaluate

$$\int_0^{\infty} \frac{dx}{(4 + x^2)(25 + x^2)}$$

10. Find the Fourier cosine transform of e^{-x^2}

11. Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms

12. Find the Fourier sine and cosine transform of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

13. Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

14. Find the function if its sine transform is $\frac{e^{-as}}{s}$ and hence deduce $F_s^{-1}\left(\frac{1}{s}\right)$

15. Find the Fourier sine transform of $f(x) = e^{-ax}$ where $a > 0$ and hence deduce that $\int_0^{\infty} \frac{s \sin(sx)}{a^2 + s^2} ds = \frac{\pi}{2} e^{-ax}$

16. Using Parseval's identity, show that $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{4a^3}$

17. Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

18. Find the Fourier sine and cosine transform of e^{-x} . Hence evaluate (i) $\int_0^{\infty} \frac{1}{(x^2 + 1)^2} dx$ and $\int_0^{\infty} \frac{x^2}{(x^2 + 1)^2} dx$