

UNIT II – FOURIER SERIES**PART –A**

1. State Dirichlet's condition (OR) State the sufficient condition for a function $f(x)$ to be expressed as a Fourier series.

ANS

- $f(x)$ is single valued, finite and periodic.
 - $f(x)$ has a finite number of finite discontinuities.
 - $f(x)$ has a finite number of maxima and minima.
 - $f(x)$ has no infinite discontinuity.
2. State whether $y=\tan x$ can be expressed as a Fourier series. If so how? If not why?

ANS

$y=\tan x$ cannot be expressed as a Fourier series, since it has infinite number of infinite discontinuity.

3. Obtain the sum at $x=1$ of the Fourier series of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}$.

ANS

Here $x=1$ is a discontinuous and middle point.

$$\text{Sum at } x=1 = \frac{\text{LHL} + \text{RHL}}{2} = \frac{1+2}{2} = \frac{3}{2}$$

4. Find the sum of the Fourier series of $f(x)=x+x^2$ in $(-\pi,\pi)$ at $x=\pi$.

ANS

Here $x=\pi$ is a discontinuous and end point.

$$\text{Sum at } x=\pi = \frac{f(-\pi) + f(\pi)}{2} = \frac{-\pi + \pi^2 + \pi + \pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$$

5. Find the constant term in the expansion of $\cos^2 x$ as a Fourier series in the interval $(-\pi,\pi)$.

ANS

Given $f(x) = \cos^2 x$

$$f(-x) = \cos^2(-x) = \cos^2(x).$$

$$f(x) + f(-x) = \cos^2 x + \cos^2 x = 2 \cos^2 x \neq 0.$$

$$f(x) - f(-x) = \cos^2 x - \cos^2 x = 0$$

$$\rightarrow f(x) \text{ is an even function } \rightarrow b_n = 0.$$

$$\begin{aligned}
 a_0 &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \cos^2 x dx \\
 &= \frac{2}{\pi} \int_0^{\pi} \frac{(1 + \cos 2x)}{2} dx \\
 &= \frac{1}{\pi} \int_0^{\pi} (1 + \cos 2x) dx \\
 &= \frac{1}{\pi} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{1}{\pi} [\pi] \\
 a_0 &= 1.
 \end{aligned}$$

$$\text{Constant Term} = \frac{a_0}{2} = \frac{1}{2}.$$

6. Obtain the first term of the Fourier series for the function $f(x) = x^2$ in $-\pi < x < \pi$.

ANS

$$\text{Given } f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2$$

$$f(x) + f(-x) = x^2 + x^2 = 2x^2 \neq 0.$$

$$f(x) - f(-x) = x^2 - x^2 = 0$$

$\rightarrow f(x)$ is an even function $\rightarrow b_n = 0$.

$$\begin{aligned}
 a_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x^2 \cdot \cos(nx) dx \\
 &= \frac{2}{\pi} \left\{ x^2 \left[\frac{\sin(nx)}{n} \right] - 2x \left[\frac{-\cos(nx)}{n^2} \right] + 2 \left[\frac{-\sin(nx)}{n^3} \right] \right\}_0^{\pi} \\
 &= \frac{2}{\pi} \left\{ \frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3} \right\}_0^{\pi} \\
 &= \frac{2}{\pi} \left\{ \frac{2\pi \cos(n\pi)}{n^2} \right\} = \frac{4(-1)^n}{n^2} \\
 \therefore a_n &= \frac{4(-1)^n}{n^2} \Rightarrow a_1 = -4
 \end{aligned}$$

7. If $x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$ in $(-\pi, \pi)$, deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

ANS

Put $x = \pi$

$$\text{Sum at } (x = \pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)$$

$x = \pi$ is a discontinuous endpoint.

$$\frac{(-\pi)^2 + \pi^2}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n$$

$$\frac{2\pi^2}{2} = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}$$

8. Give the expression for the Fourier series coefficient b_n for the function $f(x)$ defined in $(-2, 2)$.

ANS

Here $L=2$.

The expression for b_n in $(-1, 1)$ is given by $b_n = \frac{2}{2L} \int_{-L}^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$. Put $L=2$.

$$\therefore b_n = \frac{1}{2} \int_{-2}^2 f(x) \cdot \sin\left(\frac{n\pi x}{2}\right) dx.$$

9. Define Harmonic Analysis

ANS

The process of finding the Fourier series of a tabular function is called Harmonic Analysis.

10. Find the Root Mean square value of the function $f(x) = x$ in $(0, L)$.

ANS

$$\begin{aligned} \text{RMS} &= \sqrt{\frac{1}{b-a} \int_a^b f(x)^2 dx} \\ &= \sqrt{\frac{1}{L} \int_0^L x^2 dx} \\ &= \sqrt{\frac{1}{L} \left[\frac{x^3}{3} \right]_{x=0}^L} = \sqrt{\frac{1}{L} \left[\frac{L^3}{3} \right]} = \sqrt{\frac{L^2}{3}}. \end{aligned}$$

PART – B

- Find the Fourier Series of $f(x)=x^2$ in $(0,2\pi)$. Hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.
- Expand $f(x)=x(2\pi-x)$ as a Fourier Series in $(0,2\pi)$ and hence deduce that the sum of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$.
- Obtain the Fourier series of the function $f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.
- Obtain the Fourier Series of $f(x)=x+x^2$ in $(-\pi,\pi)$. Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.
- Obtain the Fourier series of the function $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$. Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.
- Find the Fourier series for $f(x)=x^2$ in $(-\pi,\pi)$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$.
- Find the Half range cosine series of $f(x)=x(\pi-x)$ in $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$.
- Find the half range cosine series of $f(x)=(x-1)^2$ in $0 < x < 1$.
- Find the Half range sine series for $f(x) = x$ in $(0,\pi)$. Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
- Find the Half range sine series of a function $f(x)=x(\pi-x)$ in $0 < x < \pi$. Hence deduce that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$.
- Find the Fourier series up to second harmonic for $y=f(x)$ from the following table.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y=f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

- Find the Fourier series up to second harmonic for the following function

x	0	1	2	3	4	5
y=f(x)	9	18	24	28	26	20

13. Compute up to second harmonic of the Fourier series of f(x) given by the following table.

x	0	T/6	T/3	T/2	2T/3	5T/6	T
y=f(x)	1.98	1.30	1.05	1.30	-0.88	-.25	1.98