

UNIT 1 – PARTIAL DIFFERENTIAL EQUATIONS

PART -A

1. Form the PDE from $(x-a)^2+(y-b)^2+z^2=r^2$.

ANS

Given $(x-a)^2+(y-b)^2+z^2=r^2$.

Diff with respect to x,

Diff with respect to y,

$$2(x-a)+2z \frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) + zp = 0 \Rightarrow (x-a) = -zp \quad 2(y-b)+2z \frac{\partial z}{\partial y} = 0 \Rightarrow (y-b) + zq = 0 \Rightarrow (y-b) = -zq$$

The pde is $(-zp)^2+(-zq)^2+z^2=r^2 \rightarrow z^2p^2+z^2q^2+z^2=r^2$.

2. Form the partial differential equation by eliminating the constants a and b from $z=(x^2+a^2)(y^2+b^2)$

ANS

Given $z=(x^2+a^2)(y^2+b^2)$

Diff with respect to x partially

Diff with respect to y partially,

$$\frac{\partial z}{\partial x} = 2x(y^2 + b^2) \Rightarrow p = 2x(y^2 + b^2) \Rightarrow (y^2 + b^2) = \frac{p}{2x} \quad \frac{\partial z}{\partial y} = 2y(x^2 + a^2) \Rightarrow q = 2y(x^2 + a^2) \Rightarrow (x^2 + a^2) = \frac{q}{2y}$$

$$z = \frac{q}{2y} \cdot \frac{p}{2x} \Rightarrow z = \frac{pq}{4xy} \text{ is the pde.}$$

3. Find the pde of the family of spheres having their centers on the Z-axis.

ANS

The equation of sphere having center at (a,b,c) with radius r is given by $(x-a)^2+(y-b)^2+(z-c)^2=r^2$.

Since center (a,b,c) lies on Z-axis, a=0 and b=0.

(1) becomes $x^2+y^2+(z-c)^2=r^2$

Diff with respect to x,

Diff with respect to y partially,

$$2x+2(z-c) \frac{\partial z}{\partial x} = 0$$

$$2y+2(z-c) \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 2x + 2(z-c)p = 0 \Rightarrow x + (z-c)p = 0 \Rightarrow (z-c) = \frac{-x}{p}$$

$$\Rightarrow 2y + 2(z-c)q = 0 \Rightarrow y + (z-c)q = 0 \Rightarrow (z-c) = \frac{-y}{q}$$

The pde is $\frac{-x}{p} = \frac{-y}{q} \Rightarrow qx = py$

4. Find the partial differential equation of all planes cutting equal intercepts from the X and Y-axis.

ANS

WKT the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (INTERCEPT FORM)

Since the required plane having equal intercepts from X and Y-axis, we have a=b.

(1) Becomes, $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$

Diff with respect to x,

Diff with respect to y,

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} + \frac{p}{c} = 0 \Rightarrow \frac{1}{a} = -\frac{p}{c}$$

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} + \frac{q}{c} = 0 \Rightarrow \frac{1}{a} = -\frac{q}{c}$$

$$-\frac{p}{c} = -\frac{q}{c} \Rightarrow p = q \text{ is the required pde.}$$

5. Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$

ANS

Given $z = f\left(\frac{y}{x}\right)$

Take $u = z$

$$v = \frac{y}{x}$$

Diff w.r.t. x partially, Diff w.r.t. x partially,

$$\frac{\partial u}{\partial x} = \frac{\partial z}{\partial x} = p$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2}$$

Diff w.r.t y partially, Diff w.r.t y partially,

$$\frac{\partial u}{\partial y} = \frac{\partial z}{\partial y} = q$$

$$\frac{\partial v}{\partial y} = \frac{1}{x}$$

The required pde is $\frac{p}{x} + \frac{qy}{x^2} = 0 \Rightarrow px + qy = 0$

6. Form the partial differential equation by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$

ANS

Given $z^2 - xy = f\left(\frac{x}{z}\right)$

Take $u = z^2 - xy$

$$v = \frac{x}{z}$$

Diff w.r.t. x partially,

Diff w.r.t. x partially,

$$\frac{\partial u}{\partial x} = 2z \frac{\partial z}{\partial x} - y = 2zp - y$$

$$\frac{\partial v}{\partial x} = \frac{z-x}{z^2} \cdot \frac{\partial z}{\partial x} = \frac{z-xp}{z^2}$$

Diff w.r.t y partially,

Diff w.r.t y partially,

$$\frac{\partial u}{\partial y} = 2z \frac{\partial z}{\partial y} - x = 2zq - x$$

$$\frac{\partial v}{\partial y} = \frac{-x}{z^2} \cdot \frac{\partial z}{\partial y} = \frac{-xq}{z^2}$$

$$(2zp - y) \cdot \frac{-xq}{z^2} - (2zq - x) \cdot \left(\frac{z-xp}{z^2} \right) = 0$$

The required pde is $\Rightarrow -xq(2zp - y) - (2zq - x)(z - xp) = 0$

$$\Rightarrow -2xzp q + xyq - 2z^2 q + 2zqx p + xz - x^2 p = 0$$

$$\Rightarrow xyq - 2z^2 q + xz - x^2 p = 0$$

7. Find the complete integral of $p+q=pq$.

ANS

The trial solution is $z=ax+by+c$ where $a+b=ab \rightarrow ab-b=a \rightarrow b(a-1)=a \Rightarrow b = \frac{a}{a-1}$.

The complete solution is $z = ax + \frac{a}{a-1}y + c$

8. Solve the partial differential equation $pq=x$.

ANS

Given $pq=x$.

Put $q=a$

$$ap=x \rightarrow p = \frac{x}{a}$$

$$dz = pdx + qdy \Rightarrow dz = \frac{x}{a} dx + a dy$$

$$\text{Integrate } \Rightarrow \int dz = \int \frac{x}{a} dx + \int a dy + b \Rightarrow z = a \cdot \log x + ay + b \text{ is the complete solution.}$$

9. Solve $(D^2 - 7DD' + 6D'^2)z=0$

ANS

The AE is $(m^2 - 7m + 6) = 0$

$$m = 6, 1.$$

$$CF = f_1(y+1x) + f_2(y+6x).$$

$$P.I = 0$$

The solution is $z=CF+PI$.

10. Solve $(D^3-2D^2D')z=0$.

ANS

The AE is $m^3-2m^2=0 \rightarrow m^2(m-2)=0 \rightarrow m=0,0,2$.

C.F. = $f_1(y+0x)+x.f_2(y+0x)+f_3(y+2x)$.

P.I.=0

The solution is $z=C.F.+P.I$.

PART - B

1. Eliminate the arbitrary function ϕ from the equation $\phi(x^2+y^2+z^2,ax+by+cz)=0$ to form a partial differential equation.
2. Eliminate the arbitrary function f and ϕ from the equation $z=f(x+ct)+g(x-ct)$ to form a partial differential equation.
3. Solve $x(y-z)p+y(z-x)q=z(x-y)$.
4. Solve $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$.
5. Solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.
6. Solve $(mz-ny)p+(nx-lz)q=(ly-mx)$.
7. Find the singular solution of $z=px+qy+\sqrt{1+p^2+q^2}$
8. Solve $z=px+qy+p^2+pq+q^2$.
9. Solve $z=px+qy+p^2q^2$
10. Solve $(D^2+2DD'+D'^2)z=e^{x+2y}+\sinh(x+y)$.
11. Solve $(D^2+3DD'-4D'^2)z=\cos(2x+y)+\sin(y)$.
12. Solve $(D^3+D^2D'-4DD'^2-4D'^3)z=\cos(2x+y)$
13. Solve $(D^3-7DD'^2-6D'^3)z=\sin(2x+y)$
14. Solve $(D^2-2DD'+D'^2)z=e^{x+2y}+\sin(2x-3y)$.
15. Solve $(D^2-4DD'+4D'^2)z=xy$.