

UNIT V – Z- TRANSFORMS AND DIFFERENCE EQUATIONS

PART -A

1. Define Z-Transform

ANS

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n).z^{-n} = F[z]$$

2. Find the Z-Transform of a^n .

ANS

$$\text{WKT } Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots = \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1} = \frac{z}{z-a}$$

3. Find Z[n]

ANS

$$\text{WKT } Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z[n] = \sum_{n=0}^{\infty} n.z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{0}{z^0} + \frac{1}{z^1} + \frac{2}{z^2} + \frac{3}{z^3} + \dots = \frac{1}{z^1} + \frac{2}{z^2} + \frac{3}{z^3} + \dots$$

$$= \frac{1}{z} \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right) = \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-2} = \frac{1}{z} \left(\frac{z}{z-1}\right)^2 = \frac{z}{(z-1)^2}$$

4. Find the Z-Transform of $\sin\left(\frac{n\pi}{2}\right)$

ANS

$$\text{w.k.t } Z[a^n . \sin(n\theta)] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + a^2}$$

$$\text{Put } a = 1 \text{ and } \theta = \frac{\pi}{2}$$

$$\therefore Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2 + 1}$$

5. Define unit step sequence. Write its z-Transform

ANS

$$u(n) = \begin{cases} 1, n \geq 0 \\ 0, \text{otherwise} \end{cases}$$

$$Z[u(n)] = \sum_{n=0}^{\infty} u(n).z^{-n} = \sum_{n=0}^{\infty} 1.z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots = \frac{z}{z-1}$$

$$\therefore Z[u(n)] = \frac{z}{z-1}$$

6. State Convolution theorem on Z-Transform

ANS

$$Z[f(n)*g(n)] = Z[f(n)].Z[g(n)] \text{ where } f(n)*g(n) = \sum_{k=0}^n f(k).g(n-k)$$

7. State and prove Initial value theorem on Z-Transform

ANS

$$\text{If } Z[f(n)] = F[z], \text{ then } f(0) = \lim_{z \rightarrow \infty} F[z]$$

Proof

$$\text{w.k.t } Z[f(n)] = \sum_{n=0}^{\infty} f(n).z^{-n} = f(0) + f(1).z^{-1} + f(2).z^{-2} + f(3).z^{-3} + \dots$$

$$F[z] = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \dots$$

$$\lim_{z \rightarrow \infty} F[z] = f(0)$$

8. Obtain $Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

ANS

$$\text{Let } F[z] = \frac{z}{(z+1)(z+2)}$$

$$z^{n-1}F[z] = \frac{z^n}{(z+1)(z+2)}$$

$z = -1$ is a simple pole. $z = -2$ is also a simple pole.

$$\text{Res}_{z=-1} = \lim_{z \rightarrow -1} \frac{z^n}{z+2} = (-1)^n.$$

$$\text{Res}_{z=-2} = \lim_{z \rightarrow -2} \frac{z^n}{z+1} = -(-2)^n.$$

$$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = (-1)^n - (-2)^n.$$

9. Form a difference equation by eliminating arbitrary constants from $u_n = A.2^{n+1}$

ANS

$$u_n = A.2^{n+1}$$

$$u_{n+1} = A.2^{n+1+1} = A.2^{n+1}.2 = 2u_n$$

The difference equation is $u_{n+1} = 2u_n$.

10. Find the difference equation generated by $y_n = a.n + b.2^n$

ANS

$$y_n = a.n + b.2^n$$

$$y_{n+1} = a.(n+1) + b.2^{n+1}$$

$$y_n = a.(n+2) + b.2^{n+2}$$

The difference equation is
$$\begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix} = 0.$$

On expanding $2ny_n + (2-3n)y_{n+1} + (n-1)y_{n+2} = 0$

11. What advantage is gained when Z-Transform is used to solve difference equation?

ANS

Z-Transform converts difference equation to algebraic equation.

12. Solve $y_{n+1}-2y_n=0$ given $y_0=3$.

ANS

Given $y_{n+1}-2y_n=0$

Apply Z-Transform on both sides, $Z[y_{n+1}]-2Z[y_n]=Z[0]$

$$z.Z[y_n]-z.y_0-2.Z[y_n]=0$$

$$(z-2).Z[y_n]-3z=0$$

$$Z[y_n] = \frac{3z}{(z-2)}$$

$$y_n = Z^{-1}\left[\frac{3z}{(z-2)}\right] = 3Z^{-1}\left[\frac{z}{(z-2)}\right] = 3.2^n$$

PART – B

1. Find the Z-transform of $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$.

2. Find $Z^{-1}\left[\frac{z(z^2 - z + 2)}{(z+1)(z-1)^2}\right]$

3. Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2 + 4)}\right]$

4. Find $Z^{-1}\left[\frac{z^3 + 3z}{(z-1)^2(z^2 + 1)}\right]$

5. Using convolution theorem, find the following

$$1. \quad z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] \quad 2. \quad z^{-1} \left[\frac{z^2}{(z+a)(z+b)} \right] \quad 3. \quad z^{-1} \left[\frac{z^2}{(z-a)^2} \right] \quad 4. \quad z^{-1} \left[\frac{z^2}{(z+a)^2} \right] \quad 5. \quad z^{-1} \left[\left(\frac{z}{z-4} \right)^3 \right]$$
$$6. \quad z^{-1} \left[\frac{8z^2}{(2z-1)(4z-1)} \right]$$

6. Find the inverse Z- Transform of $\frac{10z}{z^2 - 3z + 2}$ using Residue method.

7. Find the Inverse Z-Transform of $\frac{z(z+1)}{(z-1)^3}$ using Residue method.

8. Solve $y_{n+2} + y_n = 2$ given $y_0=0$ and $y_1=0$ by using Z-Transforms.

9. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given that $y_0=0$ and $y_1=0$.

10. Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ given that $u_0=0$ and $u_1=1$.