

**UNIT IV – FOURIER TRANSFORM**

**PART –A**

1. Define Fourier Integral theorem.

**ANS**

If  $f(x)$  is piece wise continuously differentiable and absolutely integrable in  $(-\infty, \infty)$ , then

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{is(x-t)} dt \cdot ds$$

2. Define Fourier Transform pair and its Parseval's identity.

**ANS**

◆ The Fourier Transform of  $f(x)$  is  $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx = F[s]$

◆ The inverse Fourier Transform is  $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[s] \cdot e^{-isx} ds$

◆ Parseval's identity  $\int_{-\infty}^{\infty} F[s]^2 ds = \int_{-\infty}^{\infty} f(x)^2 dx$

3. Find the Fourier Transform of  $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$

**ANS**

The Fourier transform of  $f(x)$  is  $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x} dx$$

$$F[s] = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{i(k+s)x}}{i(k+s)} \right]_{x=a}^b = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{i(k+s)b}}{i(k+s)} - \frac{e^{i(k+s)a}}{i(k+s)} \right]$$

4. State Convolution theorem on Fourier Transform

**ANS**

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)] \text{ where } f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \cdot dt$$

5. State Fourier cosine transform pair and state parseval's identity on it.

**ANS**

► The Fourier cosine transform of  $f(x)$  is  $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(sx) \cdot dx = F_c(s)$

▶ The Inverse Fourier cosine Transform is  $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_c(s) \cdot \cos(sx) \cdot ds$

▶ Parseval's identity:  $\int_0^{\infty} F_c(s)^2 ds = \int_0^{\infty} f(x)^2 dx$

6. State Fourier sine transform pair and state parseval's identity on it.

ANS

✗ The Fourier sine transform of  $f(x)$  is  $F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \sin(sx) \cdot dx = F_s(s)$

✗ The Inverse Fourier sine Transform is  $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s(s) \cdot \sin(sx) \cdot ds$

✗ Parseval's identity:  $\int_0^{\infty} F_s(s)^2 ds = \int_0^{\infty} f(x)^2 dx$

7. Find the Fourier cosine and sine transform of  $e^{-ax}$ ,  $a > 0$ .

ANS

The Fourier Cosine Transform is  $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot \cos(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \left[ \frac{a}{a^2 + s^2} \right]$

The Fourier Sine Transform is  $F_s[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \sin(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot \sin(sx) \cdot dx = \frac{2}{\sqrt{2\pi}} \left[ \frac{s}{a^2 + s^2} \right]$

8. If  $F_c[s]$  is the Fourier cosine transform of  $f(x)$ , prove that  $F_c[f(x) \cos(ax)] = \frac{1}{2} [F_c(s+a) + F_c(s-a)]$

.(OR) State and prove Modulation property

ANS

The Fourier Cosine Transform is  $F_c[f(x)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cos(sx) dx = F_c[s]$ .

$$F_c[f(x) \cdot \cos(ax)] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(ax) \cdot \cos(sx) dx$$

$$\text{WKT } \cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\begin{aligned} F_c[f(x) \cdot \cos(ax)] &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \left[ \frac{\cos(a+s)x + \cos(a-s)x}{2} \right] dx \\ &= \frac{1}{2} \left\{ \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot [\cos(a+s)x + \cos(a-s)x] dx \right\} \\ &= \frac{1}{2} \left\{ \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(s+a)x \cdot dx + \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x) \cdot \cos(s-a)x \cdot dx \right\} \end{aligned}$$

$$F_c[f(x) \cdot \cos(ax)] = \frac{1}{2} [F_c(s+a) + F_c(s-a)]$$

9. Find the function  $f(x)$  whose sine transform is  $\frac{e^{-as}}{s}$ .

**ANS**

$$\text{Given } F_s[f(x)] = \frac{e^{-as}}{s}$$

The Inverse Fourier Sine transform is  $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s[s] \cdot \sin(sx) \cdot ds$

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \cdot \sin(sx) \cdot ds$$

$$f'(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \cdot s \cdot \cos(sx) \cdot ds = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-as} \cdot \cos(sx) \cdot ds = \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2}$$

$$f'(x) = \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2} \Rightarrow f(x) = \int \frac{2}{\sqrt{2\pi}} \frac{a}{a^2 + x^2} \cdot dx = \frac{2a}{\sqrt{2\pi}} \int \frac{dx}{a^2 + x^2} = \frac{2a}{\sqrt{2\pi}} \cdot \frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore f(x) = \frac{2}{\sqrt{2\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

10. Find the function  $f(x)$  whose sine transform is  $e^{-as}$ .

**ANS**

The Inverse Fourier Sine transform is  $f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} F_s[s] \cdot \sin(sx) \cdot ds$

$$f(x) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-as} \cdot \sin(sx) \cdot ds = \frac{2}{\sqrt{2\pi}} \left[ \frac{x}{a^2 + x^2} \right]$$

**PART – B**

- Find the Fourier transform of  $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$ . Hence deduce that (i)  $\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$ . (ii)  $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ .
- Find the Fourier transform of  $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| \geq a \end{cases}$ . Hence deduce that (i)  $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ . (ii)  $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$ .
- Find the Fourier transform of  $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| \geq a \end{cases}$ . Deduce that  
 (i)  $\int_0^{\infty} \frac{\sin(s) - s \cdot \cos(s)}{s^3} \cdot \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$ . (ii)  $\int_0^{\infty} \frac{\sin(t) - t \cdot \cos(t)}{t^3} dt = \frac{\pi}{4}$ . (iii)  $\int_0^{\infty} \left[\frac{\sin(t) - t \cdot \cos(t)}{t^3}\right]^2 dt = \frac{\pi}{15}$ .
- Find the Fourier transform of  $f(x) = e^{-a|x|}$ . Deduce that  
 (i)  $\int_0^{\infty} \frac{\cos(sx)}{s^2 + a^2} ds = \frac{\pi}{2a}$  (ii)  $\int_0^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$  (iii)  $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$
- Definition of Self reciprocal.
- Find the Fourier transform of  $f(x) = e^{-a^2x^2}$ . Show that  $e^{-x^2/2}$  is self reciprocal under Fourier Transform.
- Find the Fourier Sine transform of  $f(x) = e^{-ax}$ . Deduce the inversion formula on it. Deduce that  
 $\int_0^{\infty} \frac{x^2 \cdot dx}{(x^2 + a^2)^2} = \frac{\pi}{4a}$
- Find the Fourier cosine transform of  $f(x) = e^{-a^2x^2}$ . Show that  $e^{-x^2/2}$  is self reciprocal under Fourier Cosine Transform.
- Evaluate (i)  $\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$  and (ii)  $\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)}$  using transforms.
- Find the Fourier cosine and sine transform of  $x^{n-1}$  and hence prove  $\frac{1}{\sqrt{x}}$  is self reciprocal under Fourier cosine and sine transforms.