

UNIT V DIMENSIONAL ANALYSIS AND MODEL STUDIES 9

Fundamental dimensions - dimensional homogeneity – Rayleigh's method and Buckingham Pi-Theorem - Dimensionless parameters - Similitude and model studies - Distorted Models.

Overview

Many practical flow problems of different nature can be solved by using equations and analytical procedures, as discussed in the previous modules. However, solutions of some real flow problems depend heavily on experimental data and the refinements in the analysis are made, based on the measurements. Sometimes, the experimental work in the laboratory is not only time-consuming, but also expensive. So, the dimensional analysis is an important tool that helps in correlating analytical results with experimental data for such unknown flow problems. Also, some dimensionless parameters and scaling laws can be framed in order to predict the prototype behavior from the measurements on the model. The important terms used in this module may be defined as below;

Dimensional Analysis: The systematic procedure of identifying the variables in a physical phenomena and correlating them to form a set of dimensionless group is known as *dimensional analysis*.

Dimensional Homogeneity: If an equation truly expresses a proper relationship among variables in a physical process, then it will be *dimensionally homogeneous*. The equations are correct for any system of units and consequently each group of terms in the equation must have the same dimensional representation. This is also known as the law of *dimensional homogeneity*.

Dimensional variables: These are the quantities, which actually vary during a given case and can be plotted against each other.

Dimensional constants: These are normally held constant during a given run. But, they may vary from case to case.

Pure constants: They have no dimensions, but, while performing the mathematical manipulation, they can arise.

Let us explain these terms from the following examples:

- Displacement of a free falling body is given as, $S = S_0 + V_0 t + \frac{1}{2} g t^2$, where, V_0 is the initial velocity, g is the acceleration due to gravity, t is the time, S and S_0 are the final and initial distances, respectively. Each term in this equation has the dimension of length $[L]$ and hence it is *dimensionally homogeneous*. Here, S and t are the *dimensional variables*, g , S_0 and V_0 are the *dimensional constants* and $\frac{1}{2}$ arises due to mathematical manipulation and is the *pure constant*.

- Bernoulli's equation for incompressible flow is written as, $\frac{p}{\rho} + \frac{1}{2} V^2 + gz = C$. Here, p is the pressure, V is the velocity, z is the distance, ρ is the density and g is the acceleration due to gravity. In this case, the *dimensional variables* are p, V and z , the *dimensional constants* are g, ρ and C and $\frac{1}{2}$ is the *pure constant*. Each term in this equation including the constant has dimension of $[L^2 T^{-2}]$ and hence it is *dimensionally homogeneous*.

Buckingham pi Theorem

The dimensional analysis for the experimental data of unknown flow problems leads to some non-dimensional parameters. These dimensionless products are frequently referred as *pi terms*. Based on the concept of *dimensional homogeneity*, these dimensionless parameters may be grouped and expressed in functional forms. This idea was explored by the famous scientist Edgar Buckingham (1867-1940) and the theorem is named accordingly.

Buckingham pi theorem, states that if an equation involving k variables is dimensionally homogeneous, then it can be reduced to a relationship among $(k - r)$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variable. For a physical system, involving k variables, the functional relation of variables can be written mathematically as,

$$y = f(x_1, x_2, \dots, x_k) \quad (6.1.1)$$

In Eq. (6.1.1), it should be ensured that the dimensions of the variables on the left side of the equation are equal to the dimensions of any term on the right side of equation. Now, it is possible to rearrange the above equation into a set of dimensionless products (*pi terms*), so that

$$\Pi_1 = \varphi(\Pi_2, \Pi_3, \dots, \Pi_{k-r}) \quad (6.1.2)$$

Here, $\varphi(\Pi_2, \Pi_3, \dots, \Pi_{k-r})$ is a function of Π_2 through Π_{k-r} . The required number of *pi terms* is less than the number of original reference variables by r . These reference dimensions are usually the basic dimensions M, L and T (Mass, Length and Time).

Determination of pi Terms

Several methods can be used to form dimensionless products or *pi terms* that arise in dimensional analysis. But, there is a systematic procedure called *method of repeating variables* that allows in deciding the dimensionless and independent *pi terms*. For a given problem, following distinct steps are followed.

Step I: List out all the variables that are involved in the problem. The ‘variable’ is any quantity including dimensional and non-dimensional constants in a physical situation under investigation. Typically, these variables are those that are necessary to describe the “geometry” of the system (diameter, length etc.), to define fluid properties (density, viscosity etc.) and to indicate the external effects influencing the system (force, pressure etc.). All the variables must be independent in nature so as to minimize the number of variables required to describe the complete system.

Step II: Express each variable in terms of basic dimensions. Typically, for fluid mechanics problems, the basic dimensions will be either M, L and T or F, L and T . Dimensionally, these two sets are related through Newton’s second law ($F = m.a$) so that $F = MLT^{-2}$ e.g. $\rho = ML^{-3}$ or $\rho = FL^{-4}T^2$. It should be noted that these basic dimensions should not be mixed.

Step III: Decide the required number of *pi terms*. It can be determined by using *Buckingham pi theorem* which indicates that the number of *pi terms* is equal to $(k - r)$, where k is the number of variables in the problem (determined from Step I) and r is the number of reference dimensions required to describe these variables (determined from Step II).

Step IV: Amongst the original list of variables, select those variables that can be combined to form *pi terms*. These are called as *repeating variables*. The required number of *repeating variables is equal to the number of reference dimensions*. Each *repeating variable* must be dimensionally independent of the others, i.e. they cannot be combined themselves to form any dimensionless product. Since there is a possibility of repeating variables to appear in more than one *pi term*, so dependent variables should not be chosen as one of the repeating variable.

Step V: Essentially, the *pi terms* are formed by multiplying one of the non-repeating variables by the product of the repeating variables each raised to an exponent that will make the combination dimensionless. It usually takes the form of $x_i x_1^a x_2^b x_3^c$ where the exponents *a*, *b* and *c* are determined so that the combination is dimensionless.

Step VI: Repeat the 'Step V' for each of the remaining non-repeating variables. The resulting set of *pi terms* will correspond to the required number obtained from Step III.

Step VII: After obtaining the required number of *pi terms*, make sure that all the *pi terms* are dimensionless. It can be checked by simply substituting the basic dimension (*M*, *L* and *T*) of the variables into the *pi terms*.

Step VIII: Typically, the final form of relationship among the *pi terms* can be written in the form of Eq. (6.1.2) where, Π_1 would contain the dependent variable in the numerator. The actual functional relationship among *pi terms* is determined from experiment.

Illustration of Pi Theorem

Let us consider the following example to illustrate the procedure of determining the various steps in the *pi theorem*.

Example (Pressure drop in a pipe flow)

Consider a steady flow of an incompressible Newtonian fluid through a long, smooth walled, horizontal circular pipe. It is required to measure the pressure drop per unit length of the pipe and find the number of non-dimensional parameters involved in the problem. Also, it is desired to know the functional relation among these dimensionless parameters.

Step I: Let us express all the pertinent variables involved in the experimentation of pressure drop per unit length (Δp_l) of the pipe, in the following form;

$$\Delta p_l = f(D, \rho, \mu, V) \quad (6.1.3)$$

where, D is the pipe diameter, ρ is the fluid density, μ is the viscosity of the fluid and V is the mean velocity at which the fluid is flowing through the pipe.

Step II: Next step is to express all the variables in terms of basic dimensions i.e. M , L and T . It then follows that

$$\Delta p_l = ML^{-2}T^{-2}; D = L; \rho = ML^{-3}; \mu = ML^{-1}T^{-1}; V = LT^{-1} \quad (6.1.4)$$

Step III: Apply *Buckingham theorem* to decide the number of *pi terms* required. There are five variables (including the dependent variable Δp_l) and three reference dimensions. Since, $k = 5$ and $r = 3$, only *two pi terms* are required for this problem.

Step IV: The repeating variables to form *pi terms*, need to be selected from the list D , ρ , μ and V . It is to be noted that the dependent variable should not be used as one of the repeating variable. Since, there are three reference dimensions involved, so we need to select three repeating variable. These repeating variables should be dimensionally independent, i.e. dimensionless product cannot be formed from this set. In this case, D , ρ and V may be chosen as the repeating variables.

Step V: Now, first *pi term* is formed between the dependent variable and the repeating variables. It is written as,

$$\Pi_1 = \Delta p_l D^a V^b \rho^c \quad (6.1.5)$$

Since, this combination need to be dimensionless, it follows that

$$(ML^{-2}T^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c = M^0 L^0 T^0 \quad (6.1.6)$$

The exponents a , b and c must be determined by equating the exponents for each of the terms M , L and T i.e.

$$\begin{aligned} \text{For } M: 1+c &= 0 \\ \text{For } L: -2+a+b-3c &= 0 \\ \text{For } T: -2-b &= 0 \end{aligned} \quad (6.1.7)$$

The solution of this algebraic equations gives $a = 1; b = -2; c = -1$. Therefore,

$$\Pi_1 = \frac{\Delta p_l D}{\rho V^2} \quad (6.1.8)$$

The process is repeated for remaining non-repeating variables with other additional variable (μ) so that,

$$\Pi_2 = \mu D^d \cdot V^e \cdot \rho^f \quad (6.1.9)$$

Since, this combination need to be dimensionless, it follows that

$$(ML^{-1}T^{-1})(L)^d (LT^{-1})^e (ML^{-3})^f = M^0 L^0 T^0 \quad (6.1.10)$$

Equating the exponents,

$$\begin{aligned} \text{For } M: 1+f &= 0 \\ \text{For } L: -1+d+e-3f &= 0 \\ \text{For } T: -1-e &= 0 \end{aligned} \quad (6.1.11)$$

The solution of this algebraic equation gives $d = -1; e = -1; f = -1$. Therefore,

$$\Pi_2 = \frac{\mu}{\rho V D} \quad (6.1.12)$$

Step VI: Now, the correct numbers of π terms are formed as determined in “Step III”.

In order to make sure about the dimensionality of π terms, they are written as,

$$\begin{aligned} \Pi_1 &= \frac{\Delta p_l D}{\rho V^2} = \frac{(ML^{-2}T^{-2})(L)}{(ML^{-3})(LT^{-1})^2} = M^0 L^0 T^0 \\ \Pi_2 &= \frac{\mu}{\rho V D} = \frac{(ML^{-1}T^{-1})(L)}{(ML^{-3})(LT^{-1})(L)} = M^0 L^0 T^0 \end{aligned} \quad (6.1.13)$$

Step VII: Finally, the result of dimensional analysis is expressed among the π terms

as,

$$\frac{D \Delta p_l}{\rho V^2} = \phi \left(\frac{\mu}{\rho V D} \right) = \phi \left(\frac{1}{\text{Re}} \right) \quad (6.1.14)$$

It may be noted here that Re is the Reynolds number.

Remarks

- If the difference in the number of variables for a given problem and number of reference dimensions is equal to unity, then only *one Pi* term is required to describe the phenomena. Here, the functional relationship for the *one Pi* term is a constant quantity and it is determined from the experiment.

$$\Pi_1 = \text{Constant} \quad (6.1.15)$$

- The problems involving two *Pi* terms can be described such that

$$\Pi_1 = \phi(\Pi_2) \quad (6.1.16)$$

Here, the functional relationship among the variables can then be determined by varying Π_2 and measuring the corresponding values of Π_1 .

Non Dimensional numbers in Fluid Dynamics

Forces encountered in flowing fluids include those due to inertia, viscosity, pressure, gravity, surface tension and compressibility. These forces can be written as follows;

$$\begin{aligned}
 \text{Inertia force: } m.a &= \rho V \frac{dV}{dt} \propto \rho V^2 L^2 \\
 \text{Viscous force: } \tau A &= \mu A \frac{du}{dy} \propto \mu V L \\
 \text{Pressure force: } (\Delta p)A &\propto (\Delta p) L^2 \\
 \text{Gravity force: } mg &\propto g \rho L^3 \\
 \text{Surface tension force: } \sigma L & \\
 \text{Compressibility force: } E_v A &\propto E_v L^2
 \end{aligned} \tag{6.2.1}$$

The notations used in Eq. (6.2.1) are given in subsequent paragraph of this section. It may be noted that the ratio of any two forces will be dimensionless. Since, inertia forces are very important in fluid mechanics problems, the ratio of the inertia force to each of the other forces listed above leads to fundamental dimensionless groups. Some of them are defined as given below;

Reynolds number (Re): It is defined as the ratio of inertia force to viscous force.

Mathematically,

$$\text{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu} \tag{6.2.2}$$

where V is the velocity of the flow, L is the characteristics length, ρ , μ and ν are the density, dynamic viscosity and kinematic viscosity of the fluid respectively. If Re is very small, there is an indication that the viscous forces are dominant compared to inertia forces. Such types of flows are commonly referred to as “creeping/viscous flows”. Conversely, for large Re , viscous forces are small compared to inertial effects and such flow problems are characterized as inviscid analysis. This number is also used to study the transition between the laminar and turbulent flow regimes.

Euler number (E_u) : In most of the aerodynamic model testing, the pressure data are usually expressed mathematically as,

$$E_u = \frac{\Delta p}{\frac{\rho V^2}{2}} \quad (6.2.3)$$

where Δp is the difference in local pressure and free stream pressure, V is the velocity of the flow, ρ is the density of the fluid. The denominator in Eq. (6.2.3) is called “dynamic pressure”. E_u is the ratio of pressure force to inertia force and many a times the pressure coefficient (c_p) is a also common name which is defined by same manner. In the study of cavitations phenomena, similar expressions are used where, Δp is the difference in liquid stream pressure and liquid-vapour pressure. This dimensional parameter is then called as “cavitation number”.

Froude number (F_r) : It is interpreted as the ratio of inertia force to gravity force.

Mathematically, it is written as,

$$F_r = \frac{V}{\sqrt{g \cdot L}} \quad (6.2.4)$$

where V is the velocity of the flow, L is the characteristics length descriptive of the flow field and g is the acceleration due to gravity. This number is very much significant for flows with free surface effects such as in case of open-channel flow. In such types of flows, the characteristics length is the depth of water. F_r less than unity indicates sub-critical flow and values greater than unity indicate super-critical flow. It is also used to study the flow of water around ships with resulting wave motion.

Weber number (W_e): It is defined as the ratio of the inertia force to surface tension force. Mathematically,

$$W_e = \frac{\rho V^2 L}{\sigma} \quad (6.2.5)$$

where V is the velocity of the flow, L is the characteristics length descriptive of the flow field, ρ is the density of the fluid and σ is the surface tension force. This number is taken as an index of droplet formation and flow of thin film liquids in which there is an interface between two fluids. The inertia force is dominant compared to surface tension force when, $W_e \gg 1$ (e.g. flow of water in a river).

Mach number (M) : It is the key parameter that characterizes the compressibility effects in a fluid flow and is defined as the ratio of inertia force to compressibility force. Mathematically,

$$M = \frac{V}{c} = \frac{V}{\sqrt{\frac{dp}{d\rho}}} = \frac{V}{\sqrt{\frac{E_v}{\rho}}} \quad (6.2.6)$$

where V is the velocity of the flow, c is the local sonic speed, ρ is the density of the fluid and E_v is the bulk modulus. Sometimes, the square of the Mach number is called “Cauchy number” (C_a) i.e.

$$C_a = M^2 = \frac{\rho V^2}{E_v} \quad (6.2.7)$$

Both the numbers are predominantly used in problems in which fluid compressibility is important. When, M_a is relatively small (say, less than 0.3), the inertial forces induced by fluid motion are sufficiently small to cause significant change in fluid density. So, the compressibility of the fluid can be neglected. However, this number is most commonly used parameter in compressible fluid flow problems, particularly in the field of gas dynamics and aerodynamics.

Strouhal number (S_t) : It is a dimensionless parameter that is likely to be important in unsteady, oscillating flow problems in which the frequency of oscillation is ω and is defined as,

$$S_t = \frac{\omega L}{V} \quad (6.2.8)$$

where V is the velocity of the flow and L is the characteristics length descriptive of the flow field. This number is the measure of the ratio of the inertial forces due to unsteadiness of the flow (local acceleration) to inertia forces due to changes in velocity from point to point in the flow field (convective acceleration). This type of unsteady flow develops when a fluid flows past a solid body placed in the moving stream.

In addition, there are few other dimensionless numbers that are of importance in fluid mechanics. They are listed below;

Parameter	Mathematical expression	Qualitative definition	Importance
Prandtl number	$P_r = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$E_c = \frac{V^2}{c_p T_0}$	$\frac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific heat ratio	$\gamma = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Roughness ratio	$\frac{\varepsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent rough walls
Grashof number	$G_r = \frac{\beta(\Delta T) g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{(\frac{1}{2}) \rho V^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Hydrodynamics, Aerodynamics
Lift coefficient	$C_L = \frac{L}{(\frac{1}{2}) A \rho V^2}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Hydrodynamics, Aero dynamics
Drag coefficient	$C_D = \frac{D}{(\frac{1}{2}) A \rho V^2}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Hydrodynamics, Aero dynamics

Modeling and Similitude

A “model” is a representation of a physical system which is used to predict the behavior of the system in some desired respect. The physical system for which the predictions are to be made is called “prototype”. Usually, a model is smaller than the prototype so that laboratory experiments/studies can be conducted. It is less expensive to construct and operate. However, in certain situations, models are larger than the prototype e.g. study of the motion of blood cells whose sizes are of the order of micrometers. “Similitude” is the indication of a known relationship between a model and prototype. In other words, the model tests must yield data that can be scaled to obtain the similar parameters for the prototype.

Theory of models: The dimensional analysis of a given problem can be described in terms of a set of pi terms and these non-dimensional parameters can be expressed in functional forms;

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_n) \quad (6.2.9)$$

Since this equation applies to any system, governed by same variables and if the behavior of a particular prototype is described by Eq. (6.2.9), then a similar relationship can be written for a model.

$$\Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \dots, \Pi_{nm}) \quad (6.2.10)$$

The form of the function remains the same as long as the same phenomenon is involved in both the prototype and the model. Therefore, if the model is designed and operated under following conditions,

$$\Pi_{2m} = \Pi_2 ; \Pi_{3m} = \Pi_3 \dots \dots \dots \text{and} \quad \Pi_{nm} = \Pi_n \quad (6.2.11)$$

Then it follows that

$$\Pi_1 = \Pi_{1m} \quad (6.2.12)$$

Eq. (6.2.12) is the desired “prediction equation” and indicates that the measured value of Π_{1m} obtained with the model will be equal to the corresponding Π_1 for the prototype as long as the other *pi terms* are equal. These are called “model design conditions / similarity requirements / modeling laws”.

Flow Similarity

In order to achieve similarity between model and prototype behavior, all the corresponding π terms must be equated to satisfy the following conditions.

Geometric similarity: A model and prototype are geometric similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio. In order to have geometric similarity between the model and prototype, the model and the prototype should be of the same shape, all the linear dimensions of the model can be related to corresponding dimensions of the prototype by a constant scale factor. Usually, one or more of these π terms will involve ratios of important lengths, which are purely geometrical in nature.

Kinematic similarity: The motions of two systems are kinematically similar if homogeneous particles lie at same points at same times. In a specific sense, the velocities at corresponding points are in the same direction (i.e. same streamline patterns) and are related in magnitude by a constant scale factor.

Dynamic similarity: When two flows have force distributions such that identical types of forces are parallel and are related in magnitude by a constant scale factor at all corresponding points, then the flows are dynamic similar. For a model and prototype, the dynamic similarity exists, when both of them have same length-scale ratio, time-scale ratio and force-scale (or mass-scale ratio).

In order to have complete similarity between the model and prototype, all the similarity flow conditions must be maintained. This will automatically follow if all the important variables are included in the dimensional analysis and if all the similarity requirements based on the resulting π terms are satisfied. For example, in compressible flows, the model and prototype should have same Reynolds number, Mach number and specific heat ratio etc. If the flow is incompressible (without free surface), then same Reynolds numbers for model and prototype can satisfy the complete similarity.

Model scales

In a given problem, if there are two length variables l_1 and l_2 , the resulting requirement based on the pi terms obtained from these variables is,

$$\frac{l_{1m}}{l_1} = \frac{l_{2m}}{l_2} = \lambda_l \quad (6.2.13)$$

This ratio is defined as the “length scale”. For true models, there will be only one length scale and all lengths are fixed in accordance with this scale. There are other ‘model scales’ such as velocity scale $\left(\frac{V_m}{V} = \lambda_v\right)$, density scale $\left(\frac{\rho_m}{\rho} = \lambda_\rho\right)$, viscosity scale $\left(\frac{\mu_m}{\mu} = \lambda_\mu\right)$ etc. Each of these scales needs to be defined for a given problem.

Distorted models

In order to achieve the complete dynamic similarity between geometrically similar flows, it is necessary to reproduce the independent dimensionless groups so that dependent parameters can also be duplicated (e.g. same Reynolds number between a model and prototype is ensured for dynamically similar flows).

In many model studies, dynamic similarity may also lead to incomplete similarity between the model and the prototype. If one or more of the similarity requirements are not met, e.g. in Eq. 6.2.9, if $\Pi_{2m} \neq \Pi_2$, then it follows that Eq. 6.2.12 will not be satisfied i.e. $\Pi_1 \neq \Pi_{1m}$. It is a case of distorted model for which one or more of the similar requirements are not satisfied. For example, in the study of free surface flows, both Reynolds number $\left(\frac{\rho V l}{\mu}\right)$ and Froude number $\left(\frac{V}{\sqrt{g l}}\right)$ are involved. Then,

Froude number similarity requires,

$$\frac{V_m}{\sqrt{g_m l_m}} = \frac{V}{\sqrt{g l}} \quad (6.2.14)$$

If the model and prototype are operated in the same gravitational field, then the velocity scale becomes,

$$\frac{V_m}{V} = \sqrt{\frac{l_m}{l}} = \sqrt{\lambda_l} \quad (6.2.15)$$

Reynolds number similarity requires,

$$\frac{\rho_m \cdot V_m \cdot l_m}{\mu_m} = \frac{\rho \cdot V \cdot l}{\mu} \quad (6.2.16)$$

Then, the velocity scale is,

$$\frac{V_m}{V} = \frac{\mu_m}{\mu} \cdot \frac{\rho}{\rho_m} \cdot \frac{l}{l_m} \quad (6.2.17)$$

Since, the velocity scale must be equal to the square root of the length scale, it follows that

$$\frac{V_m}{V} = \frac{(\mu/\rho)_m}{(\mu/\rho)} = \left(\frac{l}{l_m} \right)^{\frac{3}{2}} = (\lambda)^{\frac{3}{2}} \quad (6.2.18)$$

Eq. (6.2.18) requires that both model and prototype to have different kinematics viscosity scale. But practically, it is almost impossible to find a suitable fluid for the model, in small length scale. In such cases, the systems are designed on the basis of Froude number with different Reynolds number for the model and prototype where Eq. (6.2.18) need not be satisfied. Such analysis will result a “distorted model” and there are no general rules for handling distorted models, rather each problem must be considered on its own merits.