UNIT III FLOW THROUGH PIPES

Viscous flow - Shear stress, pressure gradient relationship - laminar flow between parallel plates - Laminar flow through circular tubes (Hagen poiseulle's) – Hydraulic and energy gradient - flow through pipes - Darcy -Weisbach's equation - pipe roughness -friction factor- Moody's diagram- Major and minor losses of flow in pipes - Pipes in series and in parallel.

OVERVIEW:

Being highly non-linear due to the convective acceleration terms, the *Navier-Stokes* equations are difficult to handle in a physical situation. Moreover, there are no general analytical schemes for solving the nonlinear partial differential equations. However, there are few applications where the convective acceleration vanishes due to the nature of the geometry of the flow system. So, the exact solutions are often possible. Since, the *Navier-Stokes* equations are applicable to *laminar and turbulent flow*s, the complication again arise due to fluctuations in velocity components for turbulent flow. So, these exact solutions are referred to laminar flows for which the velocity is independent of time (steady flow) or dependent on time (unsteady flow) in a well-defined manner. The solutions to these categories of the flow field can be applied to the *internal and external flows*. The flows that are bounded by walls are called as internal flows while the external flows are unconfined and free to expand. The classical example of internal flow is the pipe/duct flow while the flow over a flat plate is considered as external flow. Few classical cases of flow fields will be discussed in this module pertaining to internal and external flows.

Laminar and Turbulent Flows

The fluid flow in a duct may have three characteristics denoted as laminar, turbulent and transitional. The curves shown in Fig. 5.1.1, represents the *x*-component of the velocity as a function of time at a point 'A' in the flow. For laminar flow, there is one component of velocity $\vec{V} = u\,\hat{i}$ and random component of velocity normal to the axis becomes predominant for turbulent flows i.e. $\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$. When the flow is laminar, there are occasional disturbances that damps out quickly. The flow Reynolds number plays a vital role in deciding this characteristic. Initially, the flow may start with laminar at moderate Reynolds number. With subsequent increase in Reynolds number, the orderly flow pattern is lost and fluctuations become more predominant. When the Reynolds number crosses some limiting value, the flow is characterized as turbulent. The changeover phase is called as *transition to turbulence*. Further, if the

Reynolds number is decreased from turbulent region, then flow may come back to the laminar state. This phenomenon is known as *relaminarization*.

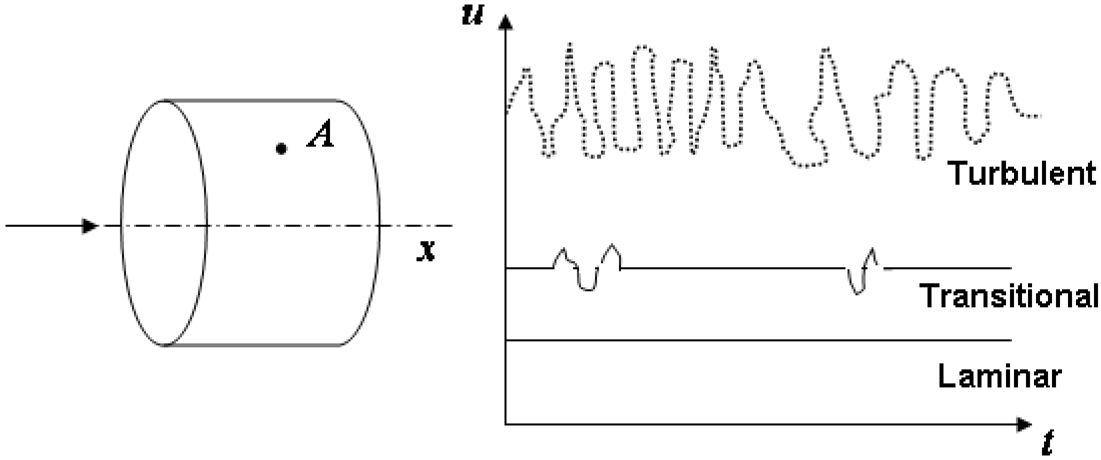

Fig. 5.1.1: Time dependent fluid velocity at a point.

The primary parameter affecting the transition is the Reynolds number defined as, $\mathrm{Re} = \frac{\rho U L}{\mu}$ where, U is the average stream velocity and L is the characteristics length/width. The flow regimes may be characterized for the following approximate ranges;

$0 < \mathrm{Re} < 1$: Highly viscous laminar motion

$1 < \mathrm{Re} < 100$: Laminar and Reynolds number dependence

$10^2 < \mathrm{Re} < 10^3$: Laminar boundary layer

$10^3 < \mathrm{Re} < 10^4$: Transition to turbulence

$10^4 < \mathrm{Re} < 10^6$: Turbulent boundary layer

$\mathrm{Re} > 10^6$: Turbulent and Reynolds number dependence

Fully Developed Flow

The fully developed steady flow in a pipe may be driven by gravity and /or pressure forces. If the pipe is held horizontal, gravity has no effect except for variation in hydrostatic pressure. The pressure difference between the two sections of the pipe, essentially drives the flow while the viscous effects provides the restraining force that exactly balances the pressure forces. This leads to the fluid moving with constant velocity (no acceleration) through the pipe. If the viscous forces are absent, then pressure will remain constant throughout except for hydrostatic variation.

In an internal flow through a long duct is shown in Fig. 5.1.2. There is an entrance *region* where the inviscid upstream flow converges and enters the tube. The viscous boundary layer grows downstream, retards the axial flow $\left[u(r,x)\right]$ at the wall and accelerates the core flow in the center by maintaining the same flow rate.

$$Q = \int u \, dA = \text{constant} \tag{5.1.1}$$

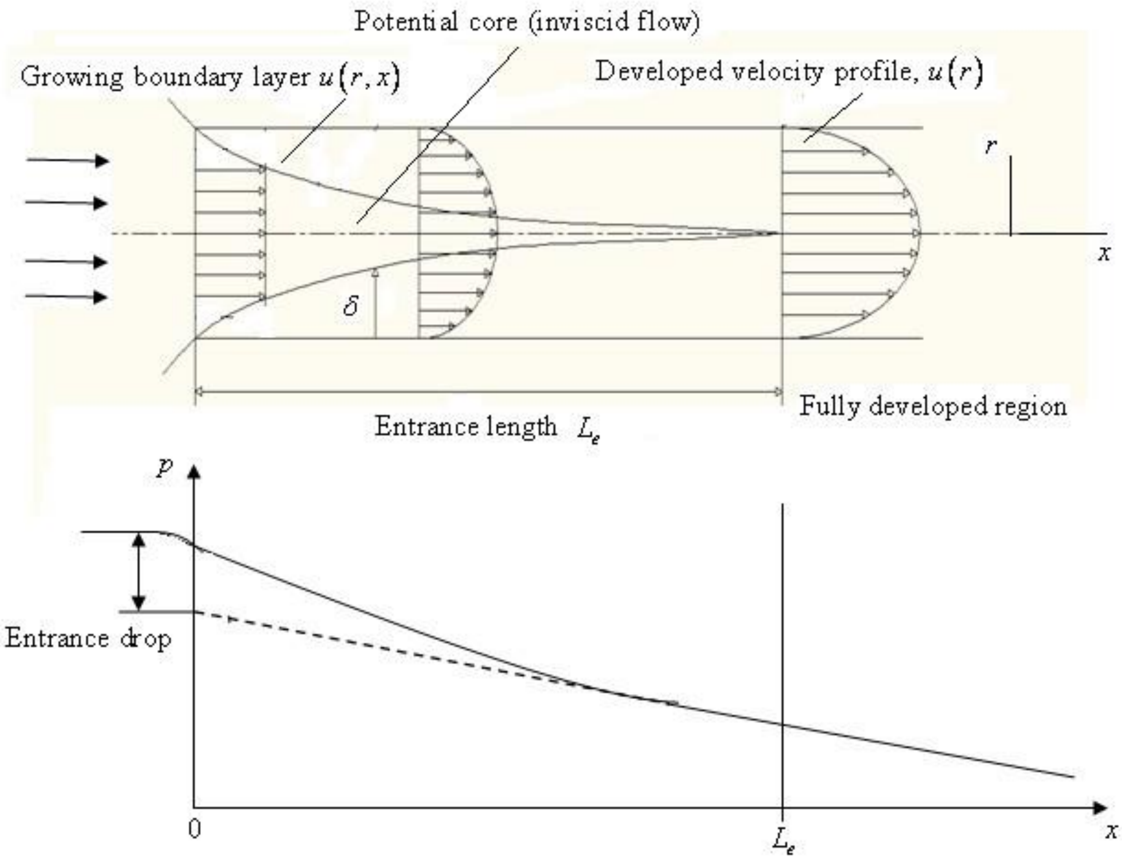

Fig. 5.1.2: Velocity profile and pressure changes in a duct flow.

At a finite distance from entrance, the boundary layers form top and bottom wall merge as shown in Fig. 5.1.2 and the inviscid core disappears, thereby making the flow entirely viscous. The axial velocity adjusts slightly till the entrance length is reached $(x = L_e)$ and the velocity profile no longer changes in x and $u \approx u(r)$ only. At this stage, the flow is said to be fully-developed for which the velocity profile and wall shear remains constant. Irrespective of laminar or turbulent flow, the pressure drops linearly with x . The typical velocity and temperature profile for laminar fully developed flow in a pipe is shown in Fig. 5.1.2. The most accepted correlations for entrance length in a flow through pipe of diameter (d) , are given below;

$$\begin{aligned}
&L_e = f(d, V, \rho, \mu); \quad V = /A \\
&\text{so that } L_e = \left(\frac{\rho Vd}{\mu}\right) = g(\mathrm{Re}) \\
&\text{Laminar flow: } \frac{L_e}{d} \approx 0.06\ \mathrm{Re} \\
&\text{Turbulent flow: } \frac{L_e}{d} \approx 4.4\ (\mathrm{Re})^{\frac{1}{6}}
\end{aligned} \tag{5.1.2}$$

Laminar and Turbulent Shear

In the absence of thermal interaction, one needs to solve continuity and momentum equation to obtain pressure and velocity fields. If the density and viscosity of the fluids is assumed to be constant, then the equations take the following form;

$$\begin{aligned}
&\text{Continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\
&\text{Momentum: } \rho \frac{d\vec{V}}{dt} = -\nabla p + \rho g + \mu \nabla^2 \vec{V}
\end{aligned} \tag{5.1.3}$$

This equation is satisfied for laminar as well as turbulent flows and needs to be solved subjected to *no-slip* condition at the wall with known inlet/exit conditions. In the case of laminar flows, there are no random fluctuations and the shear stress terms are associated with the velocity gradients terms such as, $\mu \frac{\partial u}{\partial x}$, $\mu \frac{\partial u}{\partial y}$ and $\mu \frac{\partial u}{\partial z}$ in *x*-direction. For turbulent flows, velocity and pressure varies rapidly randomly as a function of space and time as shown in Fig. 5.1.3.

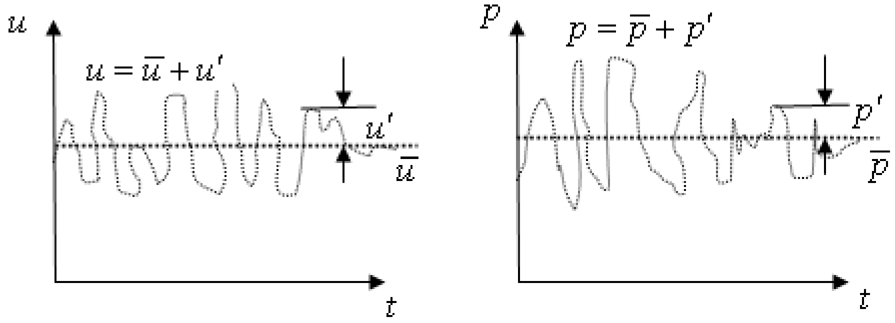

Fig. 5.1.3: Mean and fluctuating turbulent velocity and pressure.

One way to approach such problems is to define the mean/time averaged turbulent variables. The time mean of a turbulent velocity $(\bar{u})$ is defined by,

$$\bar{u} = \frac{1}{T}\int_0^T u\, dt \qquad (5.1.4)$$

where, T is the averaging period taken as sufficiently longer than the period of fluctuations. If the fluctuation $u'(= u - \bar{u})$ is taken as the deviation from its average value, then it leads to zero mean value. However, the mean squared of fluctuation $\left(\overline{u'^2}\right)$ is not zero and thus is the measure of *turbulent intensity*.

$$\overline{u'} = \frac{1}{T}\int_0^T u'\, dt = \frac{1}{T}\int_0^T (u - \bar{u})\, dt = 0; \qquad \overline{u'^2} = \frac{1}{T}\int_0^T u'^2\, dt \neq 0 \qquad (5.1.5)$$

In order to calculate the shear stresses in turbulent flow, it is necessary to know the fluctuating components of velocity. So, the *Reynolds time-averaging* concept is introduced where the velocity components and pressure are split into mean and fluctuating components;

$$u = \bar{u} + u';\;\; v = \bar{v} + v';\;\; w = \bar{w} + w';\;\; p = \bar{p} + p' \qquad (5.1.6)$$

Substitute Eq. (5.1.6) in continuity equation (Eq. 5.1.3) and take time mean of each equation;

$$\frac{1}{T}\int_0^T\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}\right)dt+\frac{1}{T}\int_0^T\left(\frac{\partial \bar{v}}{\partial y}+\frac{\partial v'}{\partial y}\right)dt+\frac{1}{T}\int_0^T\left(\frac{\partial w}{\partial z}+\frac{\partial w'}{\partial z}\right)dt=0 \qquad (5.1.7)$$

Let us consider the first term of Eq. (4.1.7),

$$\frac{1}{T}\int_0^T\left(\frac{\partial u}{\partial x}+\frac{\partial u'}{\partial x}\right)dt=\frac{1}{T}\left(\frac{\partial \bar{u}}{\partial x}\right)\int_0^T dt+\frac{1}{T}\frac{\partial}{\partial x}\int_0^T u'dt=\frac{\partial \bar{u}}{\partial x}+0=\frac{\partial \bar{u}}{\partial x} \qquad (5.1.8)$$

Considering the similar analogy for other terms, Eq. (5.1.7) is written as,

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0 \qquad (5.1.9)$$

This equation is very much similar with the continuity equation for laminar flow except the fact that the velocity components are replaced with the mean values of velocity components of turbulent flow. The momentum equation in *x*-direction takes the following form;

$$\rho\frac{d\bar{u}}{dt}=-\frac{\partial \bar{p}}{\partial x}+\rho g_x+\frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{u}}{\partial x}-\rho\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z}-\rho\overline{u'w'}\right)$$

(5.1.10)

The terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$ and $-\rho\overline{u'w'}$ in RHS of Eq. (5.1.3) have same dimensions as that of stress and called as *turbulent stresses*. For viscous flow in ducts and boundary layer flows, it has been observed that the stress terms associated with the *y*-direction (i.e. normal to the wall) is more dominant. So, necessary approximation can be made by neglecting other components of turbulent stresses and simplified expression may be obtained for Eq. (5.1.10).

$$\rho\frac{d\bar{u}}{dt}\approx-\frac{\partial \bar{p}}{\partial x}+\rho g_x+\frac{\partial \tau}{\partial y};\quad \tau=\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'}=\tau_{lam}+\tau_{tur} \qquad (5.1.11)$$

It may be noted that u' and v' are zero for laminar flows while the stress terms $-\rho\overline{u'v'}$ is positive for turbulent stresses. Hence the shear stresses in turbulent flow are always higher than laminar flow. The terms in the form of $-\rho\overline{u'v'}$, $-\rho\overline{v'w'}$, $-\rho\overline{u'w'}$ are also called as *Reynolds stresses*.

Turbulent velocity profile

A typical comparison of laminar and turbulent velocity profiles for wall turbulent flows, are shown in Fig. 5.1.4(a-b). The nature of the profile is parabolic in the case of laminar flow and the same trend is seen in the case of turbulent flow at the wall. The typical measurements across a turbulent flow near the wall have three distinct zones as shown in Fig. 5.1.4(c). The outer layer (τ_{tur}) is of two or three order magnitudes greater than the wall layer (τ_{lam}) and vice versa. Hence, the different sub-layers of Eq. (5.1.11) may be defined as follows;

- Wall layer (laminar shear dominates)
- Outer layer (turbulent shear dominates)
- Overlap layer (both types of shear are important)

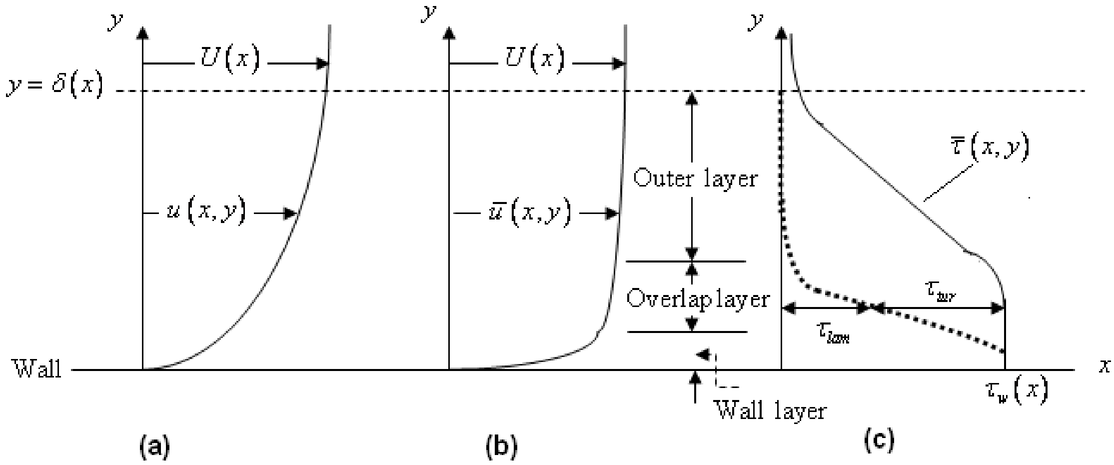

Fig 5.1.4: Velocity and shear layer distribution: (a) velocity profile in laminar flow; (b) velocity profile in turbulent flow; (c) shear layer in a turbulent flow.

In a typical turbulent flow, let the wall shear stress, thickness of outer layer and velocity at the edge of the outer layer be τ_w, δ and U, respectively. Then the velocity profiles (u) for different zones may be obtained from the empirical relations using dimensional analysis.

Wall layer: In this region, it is approximated that u is independent of shear layer thickness so that the following empirical relation holds good.

$$u = f\left(\mu, \tau_w, \rho, y\right); \quad u^+ = \frac{u}{u^*} = F\left(\frac{y\rho u^*}{\mu}\right) \text{ and } u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2} \qquad (5.1.12)$$

Eq. (5.1.12) is known as the law of wall and the quantity u^* is called as friction velocity. It should not be confused with flow velocity.

Outer layer: The velocity profile in the outer layer is approximated as the deviation from the free stream velocity and represented by an equation called as *velocity-defect law*.

$$\left(U - u\right)_{outer} = g\left(\delta, \tau_w, \rho, y\right); \quad \frac{U-u}{u^*} = G\left(\frac{y}{\delta}\right) \qquad (5.1.13)$$

Overlap layer: Most of the experimental data show the very good validation of wall law and velocity defect law in the respective regions. An intermediate layer may be obtained when the velocity profiles described by Eqs. (5.1.12 & 5.1.13) overlap smoothly. It is shown that empirically that the overlap layer varies logarithmically with y (Eq. (5.1.14). This particular layer is known as *overlap layer*.

$$\frac{u}{u^*} = \frac{1}{0.41}\ln\left(\frac{y\rho u^*}{\mu}\right) + 5 \qquad (5.1.14)$$

VISCOUS INCOMPRESSIBLE FLOW

Introduction

It has been discussed earlier that inviscid flows do not satisfy the *no-slip* condition. They slip at the wall but do not flow through wall. Because of complex nature of Navier-Stokes equation, there are practical difficulties in obtaining the analytical solutions of many viscous flow problems. Here, few classical cases of steady, laminar, viscous and incompressible flow will be considered for which the exact solution of Navier-Stokes equation is possible.

Viscous Incompressible Flow between Parallel Plates (Couette Flow)

Consider a two-dimensional incompressible, viscous, laminar flow between two parallel plates separated by certain distance $(2h)$ as shown in Fig. 5.2.1. The upper plate moves with constant velocity (V) while the lower is fixed and there is no pressure gradient. It is assumed that the plates are very wide and long so that the flow is essentially axial $(u \neq 0;\ v = w = 0)$. Further, the flow is considered far downstream from the entrance so that it can be treated as fully-developed.

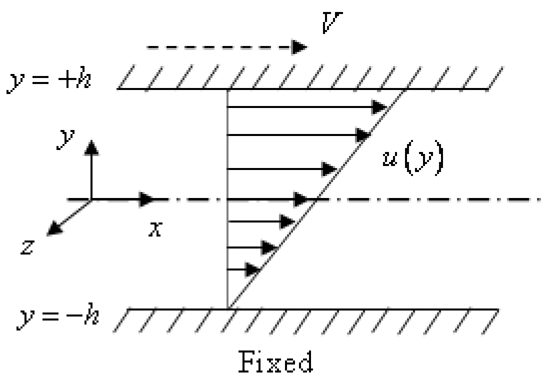

Fig. 5.2.1: Incompressible viscous flow between parallel plates with no pressure gradient.

The continuity equation is written as,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0; \quad \Rightarrow \frac{\partial u}{\partial x}=0; \quad \Rightarrow u=u(y) \text{ only} \qquad (5.2.1)$$

As it is obvious from Eq. (5.2.1), that there is only a single non-zero velocity component that varies across the channel. So, only *x*-component of Navier-Stokes equation can be considered for this planner flow.

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\text{Since } u=u(y)\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}=0;\ v=0 \qquad (5.2.2)$$

$$\text{No pressure gradient} \Rightarrow \frac{\partial p}{\partial x}=0$$

$$\text{Gravity always acts vertically downward} \Rightarrow g_x=0$$

Most of the terms in momentum equation drop out and Eq. (5.2.2) reduces to a second order ordinary differential equation. It can be integrated to obtain the solution of *u* as given below;

$$\frac{d^2u}{dy^2}=0 \quad \Rightarrow u=c_1y+c_2 \qquad (5.2.3)$$

The two constants $(c_1 \text{ and } c_2)$ can be obtained by applying no-slip condition at the upper and lower plates;

$$\text{At } y=+h; \quad u=V=c_1h+c_2$$
$$\text{At } y=-h; \quad u=0=c_1(-h)+c_2 \qquad (5.2.4)$$
$$\Rightarrow c_1=\frac{V}{2h} \text{ and } c_2=\frac{V}{2}$$

The solution for the flow between parallel plates is given below and plotted in Fig. 5.2.2 for different velocities of the upper plate.

$$u=\left(\frac{V}{2h}\right)y+\frac{V}{2} \qquad -h\le y\le +h$$
$$\text{or, } \frac{u}{V}=\frac{1}{2}\left(1+\frac{y}{h}\right) \qquad (5.2.5)$$

It is a classical case where the flow is induced by the relative motion between two parallel plates for a viscous fluid and termed as *Coutte flow*. Here, the viscosity (μ) of the fluid does not play any role in the velocity profile. The shear stress at the wall (τ_w) can be found by differentiating Eq. (5.2.5) and using the following basic equation.

$$\tau_w = \mu \frac{du}{dy} = \frac{\mu V}{2h} \quad (5.2.6)$$

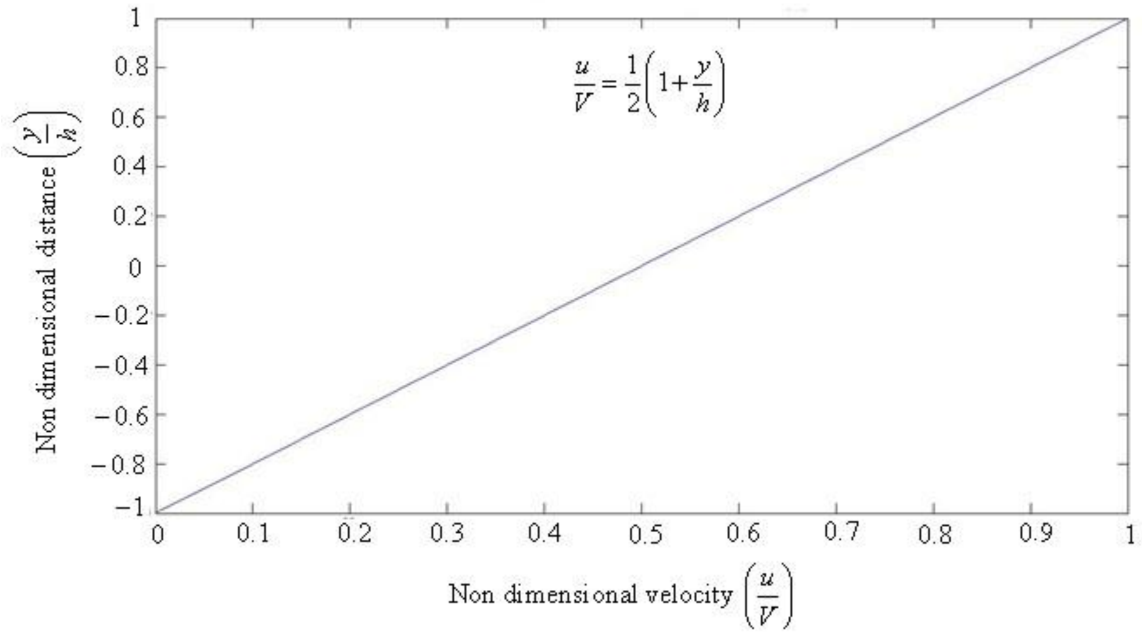

Fig. 5.2.2: Couette flow between parallel plates with no pressure gradient.

A typical application of *Couette flow* is found in the journal bearing where the main crankshaft rotates with an angular velocity (ω) and the outer one (i.e. housing) is a stationary member (Fig. 5.2.3). The gap width $(b = 2h = r_0 - r_i)$ is very small and contains lubrication oil.

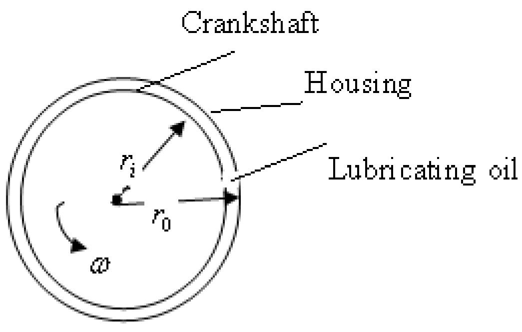

Fig. 5.2.3: Flow in a narrow gap of a journal bearing.

Since, $V = r_i \omega$, the velocity profile can be obtained from Eq. (5.2.5). The shearing stress resisting the rotation of the shaft can be simply calculated using Eq. (5.2.6).

$$\tau = \frac{\mu r_i \omega}{r_0 - r_i} \tag{5.2.7}$$

However, when the bearing is loaded (i.e. force is applied to the axis of rotation), the shaft will no longer remain concentric with the housing and the flow will no longer be parallel between the boundaries.

Viscous Incompressible Flow with Pressure Gradient (Poiseuille Flow)

Consider a two-dimensional incompressible, viscous, laminar flow between two parallel plates, separated by certain distance $(2h)$ as shown in Fig. 5.2.4. Here, both the plates are fixed but the pressure varies in *x*-direction. It is assumed that the plates are very wide and long so that the flow is essentially axial $(u \neq 0; v = w = 0)$. Further, the flow is considered far downstream from the entrance so that it can be treated as fully-developed. Using continuity equation, it leads to the same conclusion of Eq. (5.2.1) that $u = u(y)$ only. Also, $v = w = 0$ and gravity is neglected, the momentum equations in the respective direction reduces as follows;

$$\begin{aligned} &y-\text{momentum}: \frac{\partial p}{\partial y} = 0; \quad z-\text{momentum}: \frac{\partial p}{\partial z} = 0 \quad \Rightarrow p = p(x) \text{ only} \\ &x-\text{momentum}: \mu \frac{d^2u}{dy^2} = \frac{\partial p}{\partial x} = \frac{dp}{dx} \end{aligned} \tag{5.2.8}$$

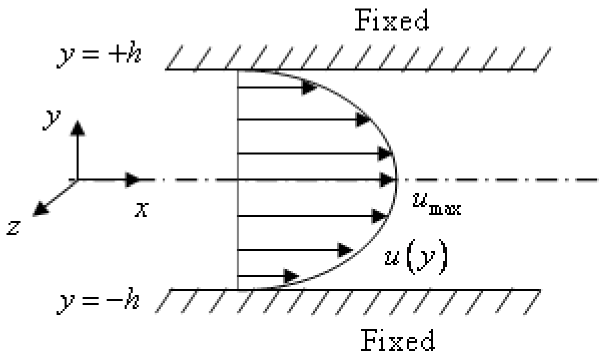

Fig. 5.2.4: Incompressible viscous flow between parallel plates with pressure gradient.

In the *x*-momentum equation, it may be noted that the left hand side contains the variation of u with y while the right hand side shows the variation of p with x. It must lead to a same constant otherwise they would not be independent to each other. Since the flow has to overcome the wall shear stress and the pressure must decrease in the direction of flow, the constant must be negative quantity. This type of pressure driven flow is called as *Poiseuille flow* which is very much common in the hydraulic systems, brakes in automobiles etc. The final form of equation obtained for a pressure gradient flow between two parallel fixed plates is given by,

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{constant} < 0 \tag{5.2.9}$$

The solution for Eq. (5.2.9) can be obtained by double integration;

$$u = \left(\frac{1}{\mu}\right)\left(\frac{dp}{dx}\right)\left(\frac{y^2}{2}\right) + c_3 y + c_4 \tag{5.2.10}$$

The constants can be found from *no-slip* condition at each wall:

$$\text{At } y = +h;\ u = 0 \Rightarrow c_1 = 0 \text{ and } c_2 = -\frac{dp}{dx}\left(\frac{h^2}{2\mu}\right) \tag{5.2.11}$$

After substitution of the constants, the general solution for Eq. (5.2.9) can be obtained;

$$u = -\left(\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1 - \frac{y^2}{h^2}\right) \tag{5.2.12}$$

The flow described by Eq. (5.2.12) forms a *Poiseuille parabola* of constant curvature and the maximum velocity (u_{max}) occurs at the centerline $y = 0$:

$$u_{max} = -\left(\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right) \tag{5.2.13}$$

The volume flow rate $(\dot{q})$ passing between the plates (per unit depth) is calculated from the relationship as follows;,

$$\dot{q} = \int_{-h}^{h} u\, dy = \int_{-h}^{h} \left(\frac{1}{2\mu}\right)\left(\frac{dp}{dx}\right)\left(h^2 - y^2\right) dy = \frac{2h^3}{3\mu}\left(\frac{dp}{dx}\right) \tag{5.2.14}$$

If Δp represents the pressure-drop between two points at a distance l along *x*-direction, then Eq. (5.2.14) is expressed as,

$$\dot{q} = \frac{2h^3}{3\mu}\left(\frac{\Delta p}{l}\right) \quad (5.2.15)$$

The average velocity $\left(u_{avg}\right)$ can be calculated as follows;

$$u_{avg} = \frac{\dot{q}}{2h} = \frac{h^2}{3\mu}\left(\frac{\Delta p}{l}\right) = \left(\frac{3}{2}\right)u_{\max} \quad (5.2.16)$$

The wall shear stress for this case can also be obtained from the definition of Newtonian fluid;

$$\tau_w = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{y=\pm h} = \mu\frac{\partial}{\partial y}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1-\frac{y^2}{h^2}\right)\right]_{y=\pm h} = \pm\left(\frac{dp}{dx}\right)h = \mp\frac{2\mu\, u_{\max}}{h} \quad (5.2.17)$$

The following silent features may be obtained from the analysis of *Couette and Poiseuille* flows;

- The *Couette* flow is induced by the relative motion between two parallel plates while the *Poiseuille* flow is a pressure driven flow.
- Both are planner flows and there is a non-zero velocity along *x*-direction while no velocity in *y and z* directions.
- The solutions for the both the flows are the exact solutions of Navier-Stokes equation.
- The velocity profile is linear for *Couette* flow with zero velocity at the lower plate with maximum velocity near to the upper plate.
- The velocity profile is parabolic for *Poiseuille* flow with zero velocity at the top and bottom plate with maximum velocity in the central line.
- In a *Poiseuille* flow, the volume flow rate is directly proportional to the pressure gradient and inversely related with the fluid viscosity.
- In a *Poiseuille* flow, the volume flow rate depends strongly on the cube of gap width.
- In a *Poiseuille* flow, the maximum velocity is 1.5-times the average velocity.

Combined Couette - Poiseuille Flow between Parallel Plates

Another simple planner flow can be developed by imposing a pressure gradient between a fixed and moving plate as shown in Fig. 5.3.1. Let the upper plate moves with constant velocity (V) and a constant pressure gradient $\left(\frac{dp}{dx}\right)$ is maintained along the direction of the flow.

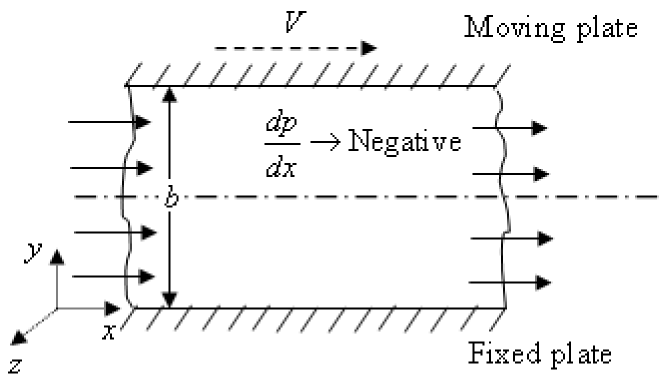

Fig. 5.3.1: Schematic representation of a combined *Couette-Poiseuille* flow.

The Navier-Stokes equation and its solution will be same as that of *Poiseuille* flow while the boundary conditions will change in this case;

$$\mu\frac{d^2u}{dy^2}=\frac{dp}{dx}=\text{constant}<0 \quad \text{and} \quad u=\left(\frac{1}{\mu}\right)\left(\frac{dp}{dx}\right)\left(\frac{y^2}{2}\right)+c_5y+c_6 \qquad (5.3.1)$$

The constants can be found with two boundary conditions at the upper plate and lower plate;

$$\begin{aligned}&\text{At } y=0;\ u=0\Rightarrow c_6=0\\&\text{At } y=b;\ u=V\Rightarrow c_5=\frac{V}{b}-\left(\frac{b}{2\mu}\right)\left(\frac{dp}{dx}\right)\end{aligned} \qquad (5.3.2)$$

After substitution of the constants, the general solution for Eq. (5.3.2) can be obtained;

$$\begin{aligned}&u=V\left(\frac{y}{b}\right)+\left(\frac{1}{2\mu}\right)\left(\frac{dp}{dx}\right)\left(y^2-by\right)\\&\text{or, } \frac{u}{V}=\left(\frac{y}{b}\right)\left[1-\left(\frac{b^2}{2\mu V}\right)\left(\frac{dp}{dx}\right)\left(1-\frac{y}{b}\right)\right]\end{aligned} \qquad (5.3.3)$$

The first part in the RHS of Eq. (5.3.3) is the solution for *Couette wall-driven flow* whereas the second part refers to the solution for *Poiseuille pressure-driven flow*. The actual velocity profile depends on the dimensionless parameter

$$P = \left(-\frac{b^2}{2\mu V} \right) \left(\frac{dp}{dx} \right) \tag{5.3.4}$$

Several velocity profiles can be drawn for different values of P as shown in Fig. 5.3.2. With $P = 0$, the simplest type of *Couette flow* is obtained with no pressure gradient. Negative values of *P* refers to *back flow* which means positive pressure gradient in the direction of flow.

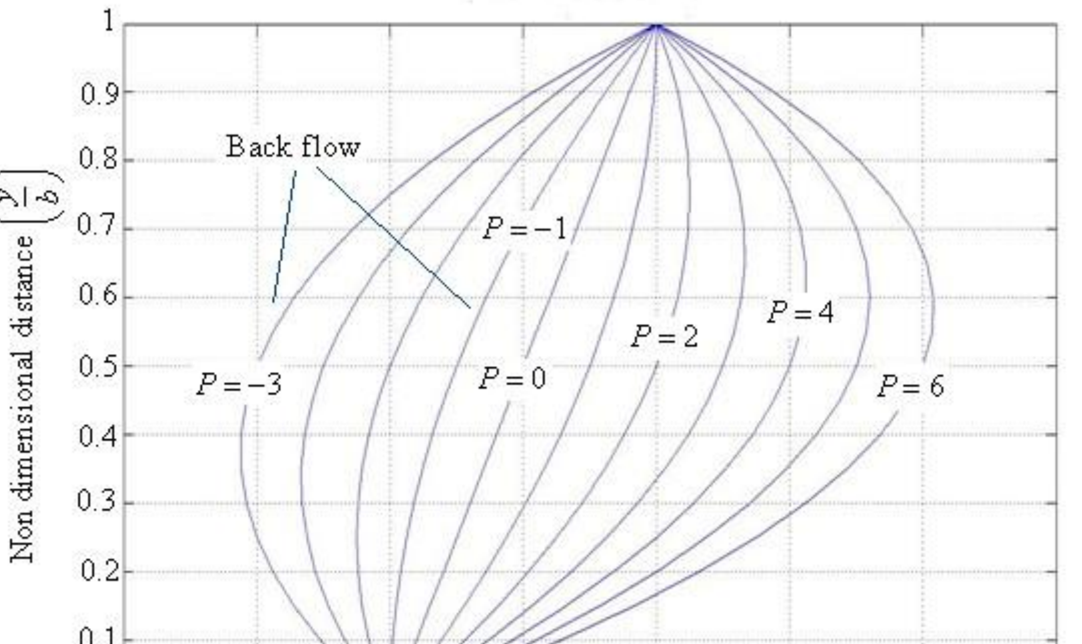

Fig. 5.3.2: Velocity profile for a combined *Couette-Poiseuille* flow between parallel plates.

Flow between Long Concentric Cylinders

Consider the flow in an annular space between two fixed, concentric cylinders as shown in Fig. 5.3.3. The fluid is having constant density and viscosity $(\mu$ and $\rho)$. The inner cylinder rotates at an angular velocity (ω_i) and the outer cylinder is fixed. There is no axial motion or end effects i.e. $v_z = 0$ and no change in velocity in the direction of θ i.e. $v_\theta = 0$. The inner and the outer cylinders have radii r_i and r_0 , respectively and the velocity varies in the direction of *r* only.

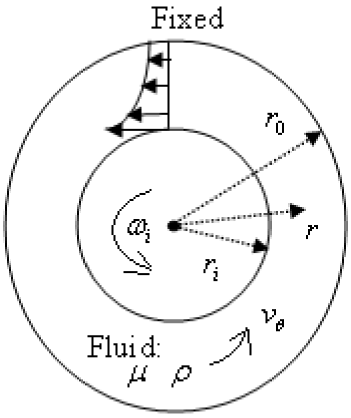

Fig. 5.3.3: Flow through an annulus.

The continuity and momentum equation may be written in cylindrical coordinates as follow;

$$\frac{1}{r}\frac{\partial(uv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}=0; \Rightarrow \frac{1}{r}\frac{d(uv_r)}{dr}=0; \Rightarrow r v_r = \text{constant} \qquad (5.3.5)$$

It is to be noted that v_θ does not vary with θ and at the inner and outer radii, there is no velocity. So, the motion can be treated as purely circumferential so that $v_r = 0$ and $v_\theta = v_\theta(r)$. The θ-momentum equation may be written as follows;

$$\rho(V\cdot\nabla)v_\theta+\frac{\rho v_r v_\theta}{r}=-\frac{1}{r}\left(\frac{\partial p}{\partial\theta}\right)+\rho g_\theta+\mu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}\right) \qquad (5.3.6)$$

Considering the nature of the present problem, most of the terms in Eq. (5.3.6) will vanish except for the last term. Finally, the basic equation for the flow between rotating cylinders becomes a linear second-order ordinary differential equation.

$$\nabla^2 v_\theta=\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right)=\frac{v_\theta}{r^2} \Rightarrow v_\theta = c_1 r+\frac{c_2}{r} \qquad (5.3.7)$$

The constants appearing in the solution of v_θ are found by no-slip conditions at the inner and outer cylinders;

$$\text{At } r = r_0;\ v_\theta = 0 = c_1 r_0 + \frac{c_2}{r_0} \quad \text{and} \quad \text{At } r = r_i;\ v_\theta = \omega_i r_i = c_1 r_i + \frac{c_2}{r_i}$$

$$\Rightarrow c_1 = \frac{\omega_i}{\left(1 - \frac{r_0^2}{r_i^2}\right)} \quad \text{and} \quad c_2 = \frac{\omega_i}{\left(\frac{1}{r_i^2} - \frac{1}{r_0^2}\right)} \tag{5.3.8}$$

The final solution for velocity distribution is given by,

$$v_\theta = \omega_i r_i \left[\frac{(r_0/r) - (r/r_0)}{(r_0/r_i) - (r_i/r_0)} \right] \tag{5.3.9}$$

FLOW IN A CIRCULAR PIPE (INTEGRAL ANALYSIS)

A classical example of a viscous incompressible flow includes the motion of a fluid in a closed conduit. It may be a *pipe* if the cross-section is round or *duct* if the conduit is having any other cross-section. The driving force for the flow may be due to the pressure gradient or gravity. In practical point of view, a *pipe/duct flow* (running in full) is driven mainly by *pressure* while an open channel flow is driven by gravity. However, the flow in a half-filled pipe having a free surface is also termed as *open channel flow*. In this section, only a fully developed laminar flow in a circular pipe is considered. Referring to geometry as shown in Fig. 5.4.1, the pipe having a radius (R) is inclined by an angle ϕ with the horizontal direction and the flow is considered in *x*-direction. The continuity relation for a steady incompressible flow in the control volume can be applied between section '1' and '2' for the constant area pipe (Fig. 5.4.1).

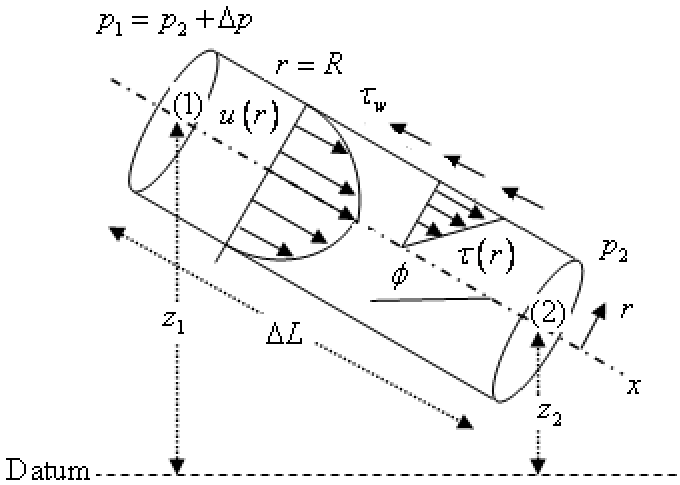

Fig. 5.4.1: Fully developed flow in an inclined pipe.

$$\int_{CS} \rho\left(V.n\right) dA = 0 \Rightarrow Q_1 = Q_2 = \text{constant} \Rightarrow u_{avg,1} = \frac{Q_1}{A_1} = u_{avg,2} = \frac{Q_2}{A_2} \qquad (5.4.1)$$

Neglect the entrance effect and assume a fully developed flow in the pipe. Since there is no shaft work or heat transfer effects, one can write the steady flow energy equation as,

$$\frac{p_1}{\rho} + \frac{1}{2}u^2_{avg,1} + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}u^2_{avg,2} + gz_2 + gh_f$$

$$\text{or, } h_f = \left(z_1 + \frac{p_1}{\rho g}\right) - \left(z_2 + \frac{p_2}{\rho g}\right) = \Delta\left(z + \frac{p}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g} \quad (5.4.2)$$

Now recall the control volume momentum relation for the steady incompressible flow,

$$\sum \vec{F} = \sum\left(\dot{m}_i \vec{V}_i\right)_{out} - \sum\left(\dot{m}_i \vec{V}_i\right)_{in} \quad (5.4.3)$$

In the present case, LHS of Eq. (5.4.3) may be considered as pressure force, gravity and shear force.

$$\Delta p\left(\pi R^2\right) + \rho g\left(\pi R^2\right)\Delta L \sin\phi - \tau_w\left(2\pi R\right)\Delta L = \dot{m}\left(V_1 - V_2\right) = 0$$

$$\text{or, } \Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g}\left(\frac{\Delta L}{R}\right) \quad \left(\Delta z = \Delta L \sin\phi\right) \quad (5.4.4)$$

$$\text{or, } \tau_w = \frac{R}{2}\left(\frac{\Delta p + \rho g\, \Delta z}{\Delta L}\right)$$

Till now, no assumption is made, whether the flow is laminar or turbulent. It can be correlated to the shear stress on the wall (τ_w). In a general sense, the wall shear stress τ_w can be assumed to be the function of flow parameters such as, average velocity (u_{avg}), fluid property $(\mu$ and $\rho)$, geometry $(d = 2R)$ and quality (roughness ε) of the pipe.

$$\tau_w = F\left(\rho, \mu, u_{avg}, d, \varepsilon\right) \quad (5.4.5)$$

By dimensional analysis, the following functional relationsip may be obtained;

$$\frac{8\tau_w}{\rho u^2_{avg}} = f = F\left(\mathrm{Re}_d, \frac{\varepsilon}{d}\right) \quad (5.4.6)$$

The desired expression for head loss in the pipe (h_f) can be obtained by combining Eqs (5.4.4 & 5.4.6).

$$h_f = f\left(\frac{L}{d}\right)\frac{u^2_{avg}}{2g} \quad (5.4.7)$$

The dimensionless parameter f is called as *Darcy friction factor* and Eq. (5.4.7) is known as *Darcy-Weisbach equation*. This equation is valid for duct flow of any cross-section, irrespective of the fact whether the flow is laminar or turbulent. In the subsequent part of this module, it will be shown that, for duct flow of any cross-section the parameter d refers to equivalent diameter and the term (ε / d) vanishes for laminar flow.

Flow in a Circular Pipe (Differential Analysis)

Let us analyze the pressure driven flow (simply *Poiseuille flow*) through a straight circular pipe of constant cross section. Irrespective of the fact that the flow is laminar or turbulent, the continuity equation in the cylindrical coordinates is written as,

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial u}{\partial x} = 0 \qquad (5.4.8)$$

The important assumptions involved in the analysis are, *fully developed flow* so that $u = u(r)$ only and there is *no swirl or circumferential variation* i.e. $\left(v_\theta = 0; \frac{\partial}{\partial \theta} = 0\right)$ as shown in Fig. 5.4.1. So, Eq. (5.4.8) takes the following form;

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \Rightarrow r v_r = \text{constant} \qquad (5.4.9)$$

Referring to Fig. 5.4.1, *no-sip* conditions should be valid at the wall $(r = R;\ v_r = 0)$. If Eq. (5.4.9) needs to be satisfied, then $v_r = 0$, everywhere in the flow field. In other words, there is only one velocity component $u = u(r)$, in a fully developed flow. Moving further to the differential momentum equation in the cylindrical coordinates,

$$\rho u \frac{\partial u}{\partial x} = -\frac{dp}{dx} + \rho g_x + \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \qquad (5.4.10)$$

Since, $u = u(r)$, the LHS of Eq. (5.4.10) vanishes while the RHS of this equation is simplified with reference to the Fig. 5.4.1.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau) = \frac{d}{dx}(p - \rho g\, x \sin\phi) = \frac{d}{dx}(p + \rho g\, z) \qquad (5.4.11)$$

It is seen from Eq. (5.4.11) that LHS varies with *r* while RHS is a function of *x* . It must be satisfied if both sides have same constants. So, it can be integrated to obtain,

$$r\tau = \frac{r^2}{2}\left[\frac{d}{dx}\left(p+\rho g z\right)\right]+c \tag{5.4.12}$$

The constant of integration (c) must be zero to satisfy the condition of no shear stress along the center line $(r = 0;\ \tau = 0)$. So, the end result becomes,

$$\tau = \frac{r}{2}\left[\frac{d}{dx}\left(p+\rho g z\right)\right] = \text{constant}\,(r) \tag{5.4.13}$$

Further, at the wall the shear stress is represented as,

$$\tau_w = \frac{R}{2}\left(\frac{\Delta p + \rho g\, \Delta z}{\Delta L}\right) \tag{5.4.14}$$

It is seen that the shear stress varies linearly from centerline to the wall irrespective of the fact that the flow is laminar or turbulent. Further, when Eqs. (5.4.4 & 5.4.14) are compared, the wall shear stress is same in both the cases.

Laminar Flow Solutions

The exact solution of Navier-Stokes equation for the steady, incompressible, laminar flow through a circular pipe of constant cross-section is commonly known as *Hagen-Poiseuille* flow. Specifically, for laminar flow, the expression for shear stress (Eq. 5.4.13) can be represented in the following form;

$$\tau = \mu\frac{du}{dr} = \frac{r}{2}K \quad \text{where } K = \frac{d}{dx}\left(p+\rho g z\right) = \text{constant}$$
$$\Rightarrow u = \frac{r^2}{2}\left(\frac{K}{\mu}\right)+c_1 \tag{5.4.15}$$

Eq. (5.4.15) can be integrated and the constant of integration is evaluated from no-slip condition, i.e. $\left(r=0;\ u=0 \Rightarrow c_1=-R^2K/4\mu\right)$. After substituting the value of c_1, Eq. (5.4.15) can be simplified to obtain the laminar velocity profile for the flow through circular pipe which is commonly known as *Hagen-Poiseuille* flow. It resembles the nature of a paraboloid falling zero at the wall and maximum at the central line (Fig. 5.4.1 and Eq. 5.4.16).

$$u=\frac{1}{4\mu}\left[-\frac{d}{dx}(p+\rho g z)\right]\left(R^2-r^2\right) \text{ and } u_{max}=\frac{R^2}{4\mu}\left[-\frac{d}{dx}(p+\rho g z)\right]$$
$$\Rightarrow \frac{u}{u_{max}}=\left(1-\frac{r^2}{R^2}\right) \tag{5.4.16}$$

The simplified form of velocity profile equation can be represented as below;

$$\frac{u}{u_{max}}=\left(1-\frac{r^2}{R^2}\right) \tag{5.4.17}$$

Many a times, the pipe is horizontal so that $\Delta z=0$ and the other results such as volume flow rate (Q) and average velocity (u_{avg}) can easily be computed.

$$Q=\int u\,dA=\int_0^R u_{max}\left(1-\frac{r^2}{R^2}\right)2\pi r\,dr=\left(\frac{u_{max}}{2}\right)\pi R^2=\frac{\pi R^4}{8\mu}\left(\frac{\Delta p}{L}\right)$$
$$\Rightarrow \Delta p=\frac{8\mu L Q}{\pi R^4}=\frac{128\mu L Q}{\pi d^4};\qquad u_{avg}=\frac{Q}{A}=\frac{Q}{\pi R^2}=\frac{u_{max}}{2} \tag{5.4.18}$$

The wall shear stress is obtained by evaluating the differential (Eq. 5.4.15) at the wall $r=R$ which is same as of Eq. (5.4.14)

$$\tau_w=\mu\left.\frac{du}{dr}\right|_{r=R}=\frac{2\mu u_{max}}{R}=\frac{R}{2}\left|\frac{d}{dx}(p+\rho g z)\right|=\frac{R}{2}\left(\frac{\Delta p+\rho g\,\Delta z}{\Delta L}\right) \tag{5.4.19}$$

Referring to Eq. (5.4.6), the laminar friction factor can be calculated as,

$$f_{lam}=\frac{8\tau_w}{\rho u_{avg}^2}=\frac{8}{\rho u_{avg}^2}\frac{R}{2}\left[\frac{d}{dx}(p+\rho g z)\right]=\frac{8}{\rho u_{avg}^2}\frac{R}{2}\left(\frac{8\mu u_{avg}}{R^2}\right)=\frac{64\mu}{\rho u_{avg} d}=\frac{64}{\mathrm{Re}_d} \tag{5.4.20}$$

The laminar head loss is then obtained from Eq. (5.4.7) as below;

$$h_{f,lam}=\left(\frac{64\mu}{\rho u_{avg} d}\right)\left(\frac{L}{d}\right)\left(\frac{u_{avg}^2}{2g}\right)=\frac{32\mu L u_{avg}}{\rho g d^2}=\frac{128\mu L Q}{\pi\rho g d^4} \tag{5.4.21}$$

The following important inferences may be drawn from the above analysis;

- The nature of velocity profile in a laminar pipe flow is paraboloid with zero at the wall and maximum at the central-line.
- The maximum velocity in a laminar pipe flow is twice that of average velocity.
- In a laminar pipe flow, the friction factor drops with increase in flow Reynolds number.
- The shear stress varies linearly from center-line to the wall, being maximum at the wall and zero at the central-line. This is true for both laminar as well as turbulent flow.
- The wall shear stress is directly proportional to the maximum velocity and independent of density because the fluid acceleration is zero.
- For a certain fluid with given flow rate, the laminar head loss in a pipe flow is directly proportional to the length of the pipe and inversely proportional to the fourth power of pipe diameter.

Turbulent Flow through Pipes

The flows are generally classified as laminar or turbulent and the turbulent flow is more prevalent in nature. It is generally observed that the turbulence in the flow field can change the mean values of any important parameter. For any geometry, the flow Reynolds number is the parameter that decides if there is any change in the nature of the flow i.e. laminar or turbulent. An experimental evidence of transition was reported first by German engineer *G.H.L Hagen* in the year 1830 by measuring the pressure drop for the water flow in a smooth pipe (Fig. 5.5.1).

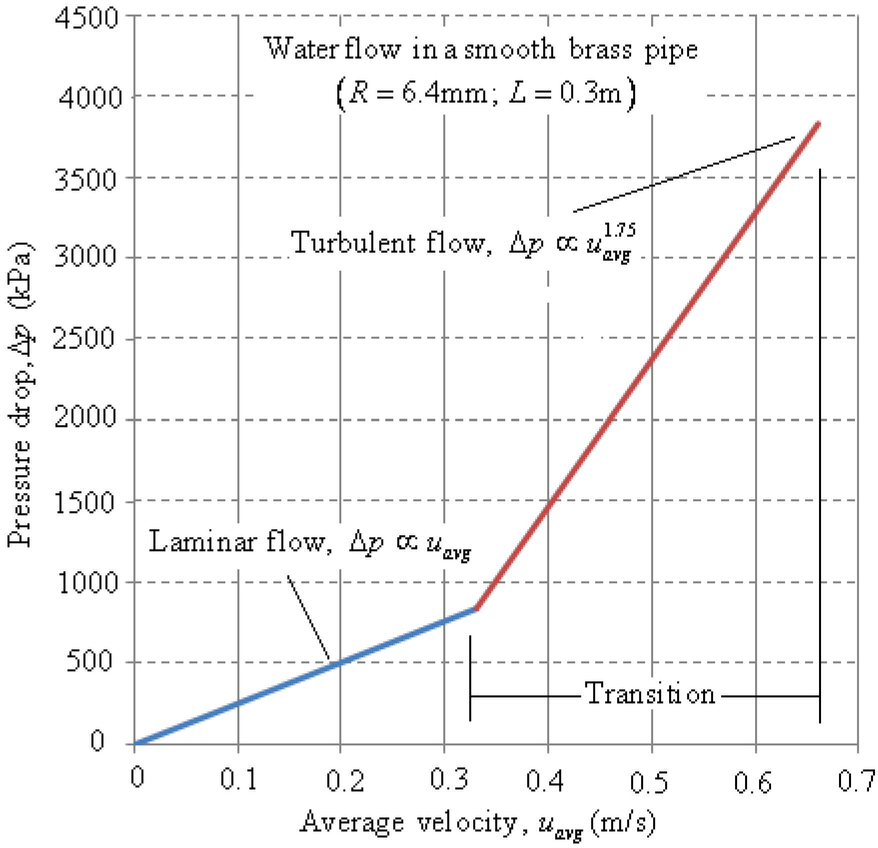

Fig. 5.5.1: Experimental evidence of transition for water flow in a brass pipe
(*Re-plotted using the data given in White 2003*)

The approximate relationship follows the pressure drop (Δp) law as given in the following equation;

$$\Delta p = \frac{\mu LQ}{R^4} + E_f \qquad (5.5.1)$$

where, E_f is the entrance effect in terms of pressure drop, μ is the fluid viscosity, Q is the volume flow, L and R are the length and radius of the pipe, respectively. It is seen from Fig. (5.5.1) that the pressure drop varies linearly with velocity up to the value 0.33m/s and a sharp change in pressure drop is observed after the velocity is increased above 0.6m/s. During the velocity range of 0.33 to 0.6m/s, the flow is treated to be under transition stage. When such a transition takes place, it is normally initiated through turbulent *spots/bursts* that slowly disappear as shown in Fig. 5.5.2. In the case of pipe flow, the flow Reynolds number based on pipe diameter is above 2100 for which the transition is noticed. The flow becomes entirely turbulent if the Reynolds number exceeds 4000.

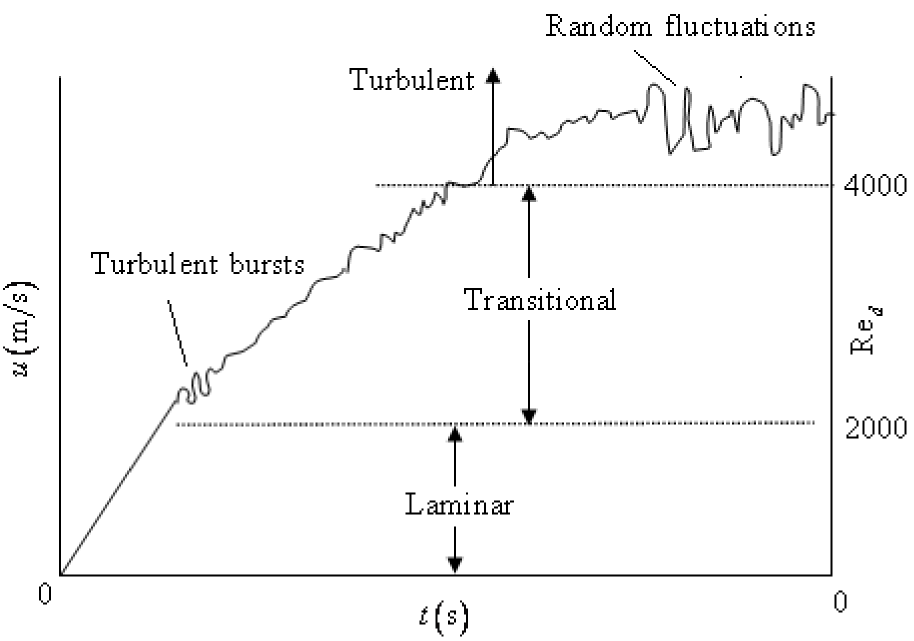

Fig. 5.5.2: Schematic representation of laminar to turbulent transition in a pipe flow.

Turbulent Flow Solutions

In the case of turbulent flow, one needs to rely on the empirical relations for velocity profile obtained from *logarithmic law*. If $u(r)$ is the local mean velocity across the pipe of radius R and $u^*\left(=\sqrt{\frac{\tau_w}{\rho}}\right)$ is the *friction velocity*, then the following empirical relation holds good;

$$\frac{u}{u^*} \approx \frac{1}{\kappa}\ln\frac{(R-r)\rho u^*}{\mu} + B \qquad (5.5.2)$$

The average velocity (u_{avg}) for this profile can be computed as,

$$u_{avg} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u^*\left[\frac{1}{\kappa}\ln\frac{(R-r)\rho u^*}{\mu} + B\right](2\pi R\,dr) = \frac{1}{2}u^*\left(\frac{2}{\kappa}\ln\frac{R\rho u^*}{\mu} + 2B - \frac{3}{\kappa}\right)$$

(5.5.3)

Using the approximate values of $\kappa = 0.41$ and $B = 5$, the simplified relation for turbulent velocity profile is obtained as below;

$$\frac{u_{avg}}{u^*} \approx 2.44\ln\left(\frac{R\rho u^*}{\mu}\right) + 1.34 \qquad (5.5.4)$$

Recall the *Darcy friction factor* which relates the wall shear stress (τ_w) and average velocity (u_{avg});

$$f = \frac{8\tau_w}{\rho u_{avg}^2} \Rightarrow u_{avg} = \left(\frac{8\tau_w}{\rho f}\right)^{1/2} = u^*\left(\frac{8}{f}\right)^{1/2} \Rightarrow \frac{u_{avg}}{u^*} = \left(\frac{8}{f}\right)^{1/2} \qquad (5.5.5)$$

Rearrange the first term appearing in RHS of Eq. (5.5.4)

$$\frac{R\rho u^*}{\mu} = \frac{(1/2)\,d\,u_{avg}\,\rho}{\mu}\left(\frac{u^*}{u_{avg}}\right) = \frac{1}{2}\mathrm{Re}_d\left(\frac{f}{8}\right)^{1/2} \qquad (5.5.6)$$

Substituting Eqs. (5.5.5 & 5.5.6) in Eq. (5.5.4) and simplifying, one can get the following relation for friction factor for the turbulent pipe flow.

$$\frac{1}{f^{0.5}} \approx 1.99\log\left(\mathrm{Re}_d\, f^{0.5}\right) - 1.02 \qquad (5.5.7)$$

Since, Eq. (5.5.7) is implicit in nature, it becomes cumbersome to obtain friction factor for a given Reynolds number. So, there are many alternative explicit approximations as given below;

$$f = 0.316\left(\mathrm{Re}_d\right)^{-0.25} \quad 4000 < \mathrm{Re}_d < 10^5$$
$$= \left(1.8\log\frac{\mathrm{Re}_d}{6.9}\right)^{-2} \tag{5.5.8}$$

Further, the maximum velocity in the turbulent pipe flow is obtained from (5.5.2) and is evaluated at $r = 0$;

$$\frac{u_{max}}{u^*} \approx \frac{1}{\kappa}\ln\frac{R\rho u^*}{\mu} + B \tag{5.5.9}$$

Another correlation may be obtained by relating Eq. (5.5.9) with the average velocity (Eq. 5.5.3);

$$\frac{u_{avg}}{u_{max}} \approx \left(1 + 1.33\sqrt{f}\right)^{-1} \tag{5.5.10}$$

For a horizontal pipe at low Reynolds number, the head loss due to friction can be obtained from pressure drop as shown below;

$$h_{f,tur} = \frac{\Delta p}{\rho g} = f\left(\frac{L}{d}\right)\left(\frac{u_{avg}^2}{2g}\right) \approx 0.316\left(\frac{\mu}{\rho u_{avg} d}\right)^{0.25}\left(\frac{L}{d}\right)\left(\frac{u_{avg}^2}{2g}\right)$$
$$\approx 0.316\left(\frac{1}{\mathrm{Re}_d}\right)^{0.25}\left(\frac{L}{d}\right)\left(\frac{u_{avg}^2}{2g}\right) \tag{5.5.11}$$

Simplifying Eq. (5.4.11), the pressure drop in a turbulent pipe flow may be expressed in terms of average velocity or flow rate;

$$\Delta p \approx 0.158 L\rho^{0.75}\mu^{0.25}d^{-1.25}u_{avg}^{1.75} \approx 0.241 L\rho^{0.75}\mu^{0.25}d^{-4.75}Q^{1.75} \tag{5.5.12}$$

For a given pipe, the pressure drop increases with average velocity power of 1.75 (Fig. 5.5.1) and varies slightly with the viscosity which is the characteristics of a turbulent flow. Again for a given flow rate, the turbulent pressure drop decreases with diameter more sharply than the laminar flow formula. Hence, the simplest way to reduce the pumping pressure is to increase the size of the pipe although the larger pipe is more expensive.

Moody Chart

The *surface roughness* is one of the important parameter for initiating transition in a flow. However, its effect is negligible if the flow is laminar but a turbulent flow is strongly affected by roughness. The surface roughness is related to frictional resistance by a parameter called as roughness ratio (ε/d), where ε is the roughness height and d is the diameter of the pipe. The experimental evidence show that friction factor (f) becomes constant at high Reynolds number for any given roughness ratio (Fig. 5.5.3). Since a turbulent boundary layer has three distinct regions, the friction factor becomes more dominant at low/moderate Reynolds numbers. So another dimensionless parameter $\varepsilon^{+}\left(=\frac{\varepsilon u_{*}\rho}{\mu}\right)$, is defined that essentially show the effects of surface roughness on friction at low/moderate Reynolds number. In a hydraulically smooth wall, there is no effect of roughness on friction and for a fully rough flow, the sub-layer is broken and friction becomes independent of Reynolds number.

$\varepsilon^{+} < 5$: Hydraulically smooth wall

$5 < \varepsilon^{+} < 70$:Transitional roughness

$\varepsilon^{+} > 70$: Fully rough flow

The dependence of friction factor on roughness ratio and Reynolds number for a turbulent pipe flow is represented by *Moody chart*. It is an accepted design formula for turbulent pipe friction within an accuracy of ±15% and based on the following empirical relations;

$$\frac{1}{f^{0.5}} = -2\log\left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}_d\, f^{0.5}}\right); \quad \frac{1}{f^{0.5}} = -1.8\log\left[\left(\frac{\varepsilon/d}{3.7}\right)^{1.11} + \frac{6.9}{\mathrm{Re}_d}\right] \qquad (5.5.13)$$

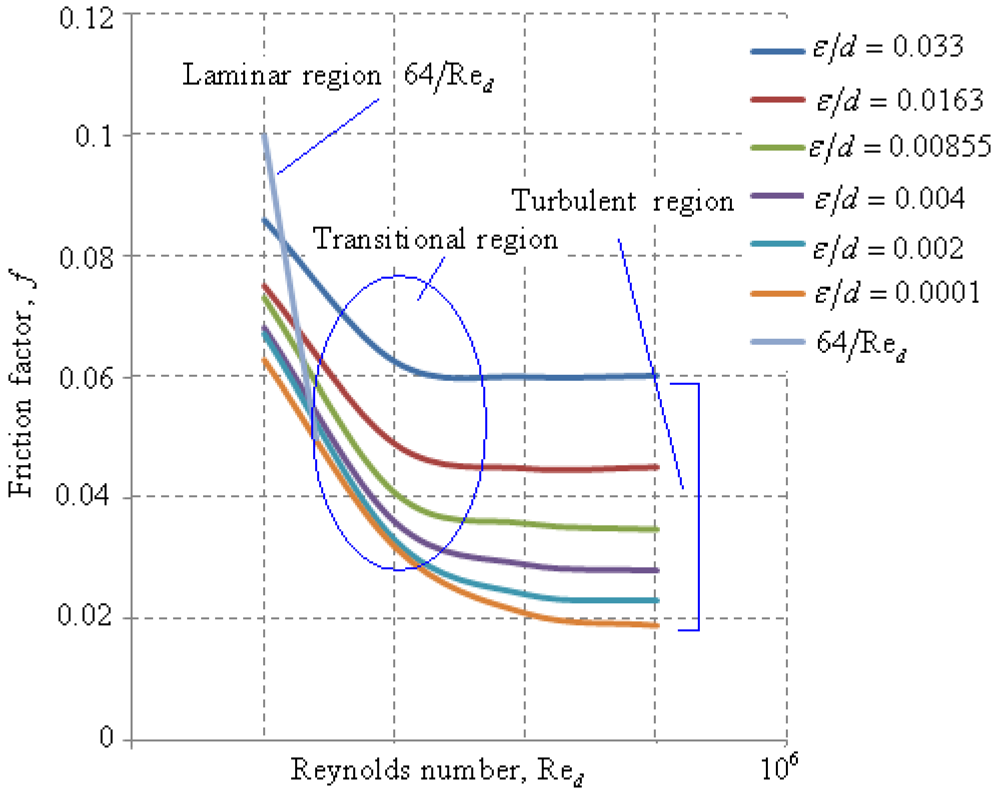

Fig. 5.5.3: Effect of wall roughness on turbulent pipe flow.
(*Re-plotted using the data given in White 2003*)

Non-Circular Ducts and Hydraulic Diameter

The analysis of fully-developed flow (laminar/turbulent) in a non-circular duct is more complicated algebraically. The concept of *hydraulic diameter* is a reasonable method by which one can correlate the laminar/turbulent fully-developed pipe flow solutions to obtain approximate solutions of non-circular ducts. As derived from momentum equation in previous section, the head loss (h_f) for a pipe and the wall shear stress (τ_w) is related as,

$$h_f = f\left(\frac{L}{d}\right)\left(\frac{u_{avg}^2}{2g}\right) = \frac{2\tau_w}{\rho g}\frac{\Delta L}{R} \qquad (5.6.1)$$

The analogous form of same equation for a non-circular duct is written as,

$$h_f = \frac{2\bar{\tau}_w}{\rho g}\frac{\Delta L}{\left(A/P_e\right)} \qquad (5.6.2)$$

where, $\bar{\tau}_w$ is the average shear stress integrated around the perimeter of the non-circular duct (P_e) so that the ratio of cross-sectional area (A) and the perimeter takes the form of length scale similar to the pipe radius (R) . So, the hydraulic radius (R_h) of a non-circular duct is defined as,

$$R_h = \frac{A}{P_e} = \frac{\text{Cross-sectional area}}{\text{Wetted perimeter}} \qquad (5.6.3)$$

If the cross-section is circular, the hydraulic diameter can be obtained from Eq. (5.6.3) as, $d_h = 4R_h$. So, the corresponding parameters such as friction factor and head loss for non-circular ducts (NCD) are then written as,

$$f_{NCD} = \frac{8\bar{\tau}_w}{\rho u_{avg}^2};\ h_f = f_{NCD}\left(\frac{L}{4R_h}\right)\frac{u_{avg}^2}{2g} \qquad (5.6.4)$$

It is to be noted that the wetted perimeter includes all the surfaces acted upon by the shear stress. While finding the laminar/turbulent solutions of non-circular ducts, one must replace the radius/diameter of pipe flow solutions with the length scale term of hydraulic radius/diameter.

Minor Losses in Pipe Systems

The fluid in a typical piping system consists of inlets, exits, enlargements, contractions, various fittings, bends and elbows etc. These components interrupt the smooth flow of fluid and cause additional losses because of mixing and flow separation. So in typical systems with long pipes, the total losses involve the *major losses* (head loss contribution) and the *minor losses* (any other losses except head loss). The major head losses for laminar and turbulent pipe flows have already been discussed while the cause of additional minor losses may be due to the followings;

- Pipe entrance or exit
- Sudden expansion or contraction
- Gradual expansion or contraction
- Losses due to pipe fittings (valves, bends, elbows etc.)

A desirable method to express minor losses is to introduce an equivalent length $\left(L_{eq}\right)$ of a straight pipe that satisfies *Darcy friction-factor* relation in the following form;

$$h_m = f\left(\frac{L_{eq}}{d}\right)\left(\frac{u_{avg}^2}{2g}\right) = K_m\left(\frac{u_{avg}^2}{2g}\right); \quad L_{eq} = \frac{K_m d}{f} \qquad (5.6.5)$$

where, K_m is the minor loss coefficient resulting from any of the above sources. So the total loss coefficient for a constant diameter (d) pipe is given by the following expression;

$$\Delta h^{total} = h^f + \sum h^m = \left(\frac{u_{avg}^2}{2g}\right)\left(\frac{fL}{d} + \sum K_m\right) \qquad (5.6.6)$$

It should be noted from Eq. (5.6.6) that the losses must be added separately if the pipe size and the average velocity for each component change. The total length $\left(L\right)$ is considered along the pipe axis including any bends.

Entrance and Exit Losses: Any fluid from a reservoir may enter into the pipe through variety of shaped region such as re-entrant, square-edged inlet and rounded inlet. Each of the geometries shown in Fig. 5.6.1 is associated with a minor head loss coefficient (K_m) . A typical flow pattern (Fig. 5.6.2) of a square-edged entrance region has a *vena-contracta* because the fluid cannot turn at right angle and it must separate from the sharp corner. The maximum velocity at the section (2) is greater than that of section (3) while the pressure is lower. Had the flow been slowed down efficiently, the kinetic energy could have converted into pressure and an ideal pressure distribution would result as shown through dotted line (Fig. 5.6.2). An obvious way to reduce the entrance loss is to rounded entrance region and thereby reducing the *vena-contracta* effect.

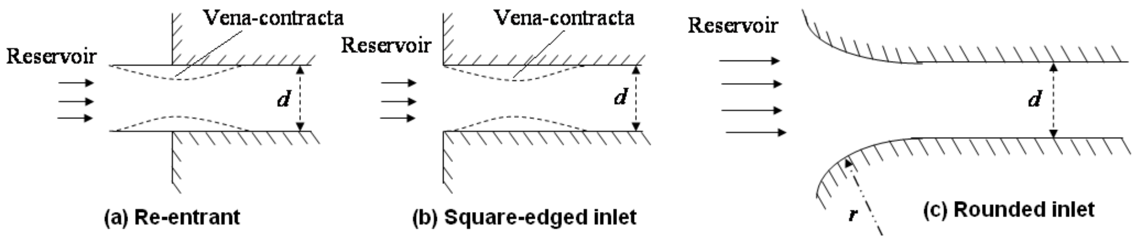

Fig. 5.6.1: Typical inlets for entrance loss in a pipe: (a) Reentrant $(K_m = 0.8)$ **; (b) Sharp-edged inlet** $(K_m = 0.5)$ **; (c) Rounded inlet** $(K_m = 0.04)$.

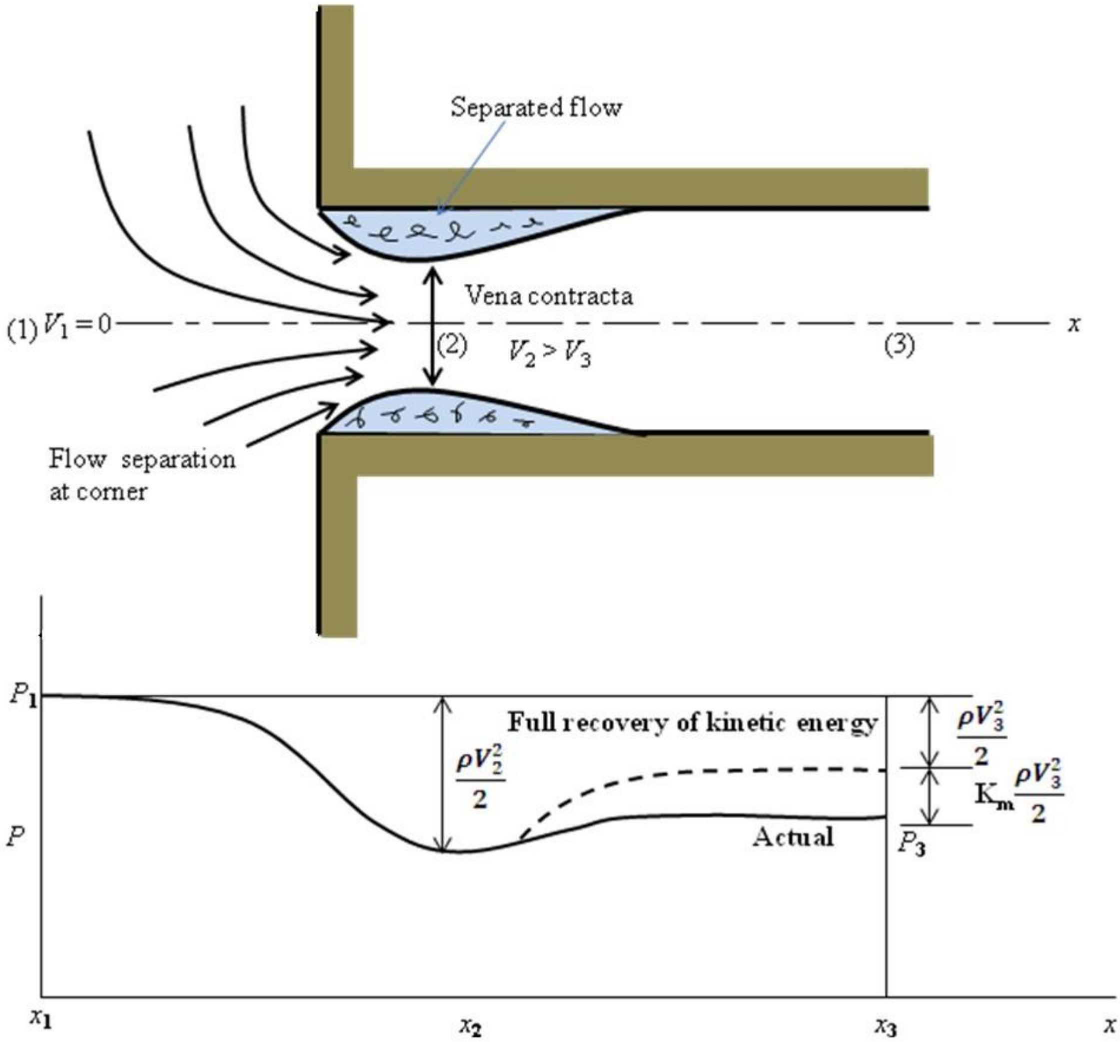

Fig. 5.6.2: Flow pattern for sharp-edged entrance.

The minor head loss is also produced when the fluid flows through these geometries enter into the reservoir (Fig. 5.6.1). These losses are known as exit losses. In these cases, the flow simply passes out of the pipe into the large downstream reservoir, loses its entire velocity head due to viscous dissipation and eventually comes to rest. So, the minor exit loss is equivalent to one velocity head $(K_m = 1)$, no matter how well the geometry is rounded.

Sudden Expansion and Contraction: The minor losses also appear when the flow through the pipe takes place from a larger diameter to the smaller one or vice versa. In the case of sudden expansion, the fluid leaving from the smaller pipe forms a jet initially in the larger diameter pipe, subsequently dispersed across the pipe and a fully-developed flow region is established (Fig. 5.6.3). In this process, a portion of the kinetic energy is dissipated as a result of viscous effects with a limiting case $(A_1/A_2)=0$.

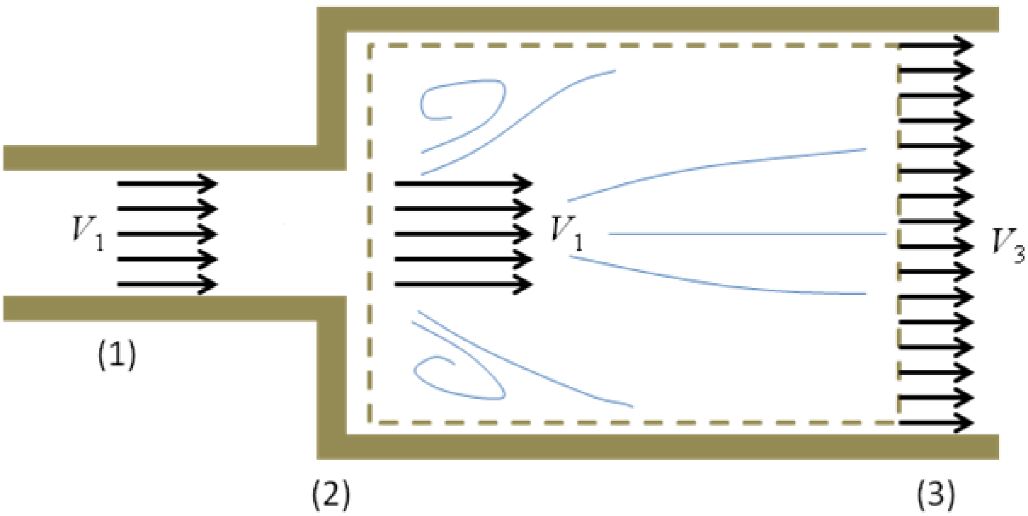

Fig. 5.6.3: Flow pattern during sudden expansion.

The loss coefficient during sudden expansion can be obtained by writing control volume continuity and momentum equation as shown in Fig. 5.6.3. Further the energy equation is applied between the sections (2) and (3). The resulting governing equations are written as follows;

$$\begin{aligned} &\text{Continuity}: \; A_1 V_1 = A_2 V_2 \\ &\text{Momentum}: \; p_1 A_3 - p_3 A_3 = \rho A_3 V_3 (V_3 - V_1) \qquad (5.6.7) \\ &\text{Energy}: \; \frac{p_1}{\rho g} + \frac{V_1^2}{2g} = \frac{p_3}{\rho g} + \frac{V_3^2}{2g} + h_m \end{aligned}$$

The terms in the above equation can be rearranged to obtain the loss coefficient as given below;

$$K_m = \frac{h_m}{\left(V_1^2/2g\right)} = \left(1 - \frac{A_1}{A_2}\right)^2 = \left(1 - \frac{d^2}{D^2}\right)^2 \qquad (5.6.8)$$

Here, A_1 and A_2 are the cross-sectional areas of small pipe and larger pipe, respectively. Similarly, d and D are the diameters of small and larger pipe, respectively.

For the case of sudden contraction, the flow initiates from a larger pipe and enters into the smaller pipe (Fig. 5.6.4). The flow separation in the downstream pipe causes the main stream to contract through minimum diameter (d_{min}), called as *vena-contracta*. This is similar to the case as shown in Fig. 5.6.2. The value of minor loss coefficient changes gradually (Fig. 5.6.5) from one extreme with $K_m = 0.5$ at $(A_1 / A_2) = 0$ to the other extreme of $K_m = 0$ at $(A_1 / A_2) = 1$. Another empirical relation for minor loss coefficient during sudden contraction is obtained through experimental evidence (Eq. 5.6.9) and it holds good with reasonable accuracy in many practical situations.

$$K_m \approx 0.42\left(1 - \frac{d^2}{D^2}\right) \tag{5.6.9}$$

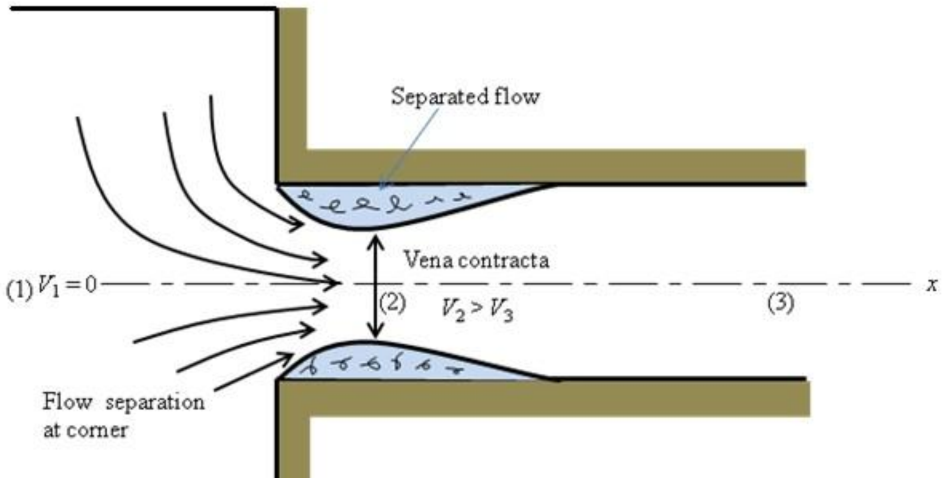

Fig. 5.6.4: Flow pattern during sudden contraction.

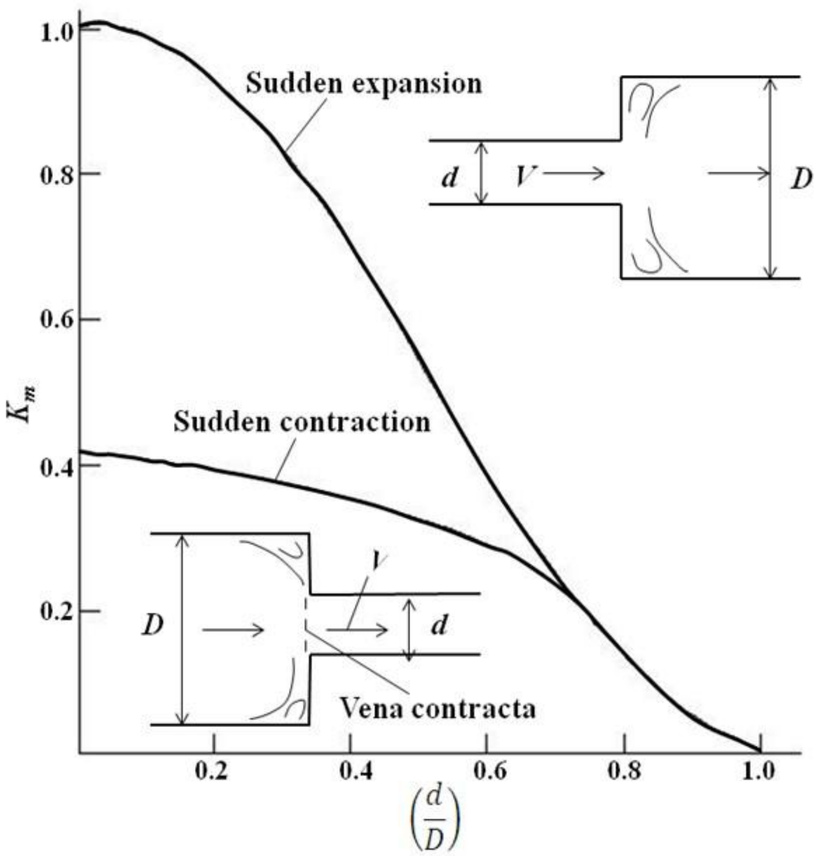

Fig. 5.6.5: Variation of loss coefficient with area ratio in a pipe.

<u>Gradual Expansion and Contraction</u>: If the expansion or contraction is gradual, the losses are quite different. A gradual expansion situation is encountered in the case of a diffuser as shown in Fig. 5.6.6. A diffuser is intended to raise the static pressure of the flow and the extent to which the pressure is recovered, is defined by the parameter pressure-recovery coefficient $\left(C_p\right)$. The loss coefficient is then related to this parameter C_p. For a given area ratio, the higher value of C_p implies lower loss coefficient K_m.

$$C_p = \frac{p_2 - p_1}{(1/2)\rho V_1^2}; \quad K_m = \frac{h_m}{V_1^2/(2g)} = 1 - \left(\frac{d_1}{d_2}\right)^4 - C_p \qquad (5.6.10)$$

When the contraction is gradual, the loss coefficients based on downstream velocities are very small. The typical values of K_m range from 0.02 – 0.07 when the included angle changes from 30º to 60º. Thus, it is relatively easy to accelerate the fluid efficiently.

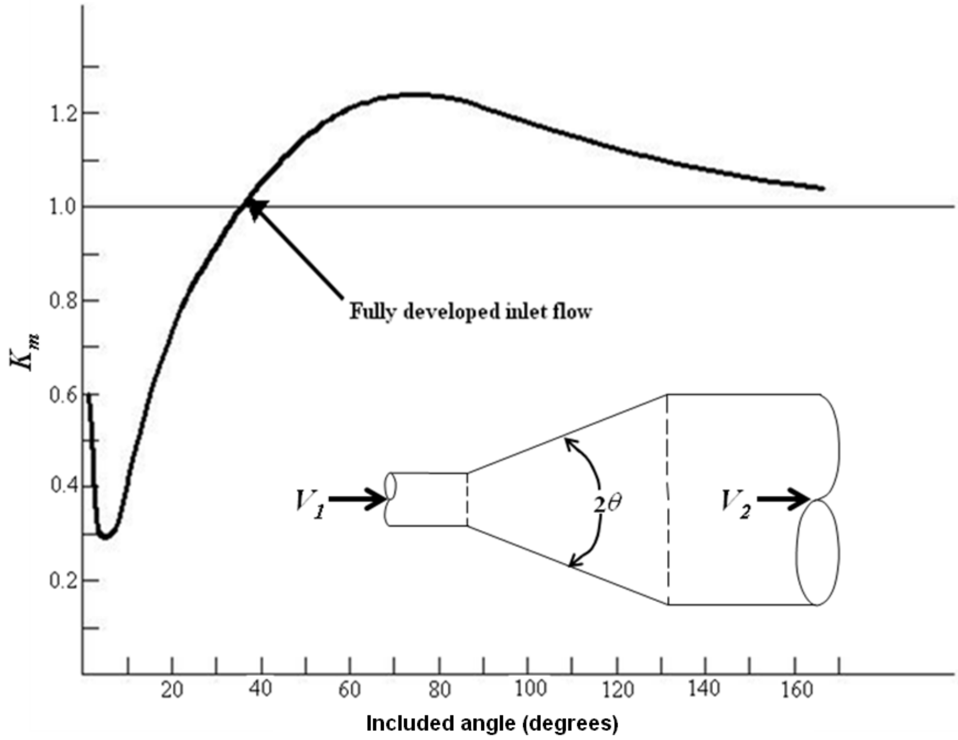

Fig. 5.6.6: Loss coefficients for gradual expansion and contraction.

Minor losses due to pipe fittings: A piping system components normally consists of various types of fitting such as valves, elbows, tees, bends, joints etc. The loss coefficients in these cases strongly depend on the shape of the components. Many a times, the value of K_m is generally supplied by the manufacturers. The typical values may be found in any reference book