Duality in LPP

Every LPP called the **primal** is associated with another LPP called **dual**. Either of the problems is primal with the other one as dual. The optimal solution of either problem reveals the information about the optimal solution of the other.

Let the primal problem be

Max $Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_n$

and

$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$

The corresponding dual is defined as

Min $Z_w = b_1w_1 + b_2w_2 + \dots + b_mw_m$

Subject to restrictions

$a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \geq c_1$

$a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \geq c_2$

.

.

.

$a_{1n}w_1 + a_{2n}w_2 + \dots\dots\dots + a_{mn}w_m \geq c_n$

and

$w_1, w_2, \dots, w_m \geq 0$

Matrix Notation

Primal

Max $Z_x = CX$

Subject to

$AX \leq b$ and $X \geq 0$

Dual

Min $Z_w = b^T W$

Subject to

$A^T W \geq C^T$ and $W \geq 0$

Important characteristics of Duality

1. Dual of dual is primal
2. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
3. If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
4. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.
5. If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.
6. If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

Steps for a Standard Primal Form

Step 1 – Change the objective function to Maximization form

Step 2 – If the constraints have an inequality sign '≥' then multiply both sides by -1 and convert the inequality sign to '≤'.

Step 3 – If the constraint has an '=' sign then replace it by two constraints involving the inequalities going in opposite directions. For example $x_1 + 2x_2 = 4$ is written as

$x_1+2x_2 \leq 4$

$x_1+2x_2 \geq 4$ (using step2) → $-x_1-2x_2 \leq -4$

Step 4 – Every unrestricted variable is replaced by the difference of two non-negative variables.

Step5 – We get the standard primal form of the given LPP in which.

- All constraints have '≤' sign, where the objective function is of maximization form.
- All constraints have '≥' sign, where the objective function is of minimization from.

Rules for Converting any Primal into its Dual

1. Transpose the rows and columns of the constraint co-efficient.
2. Transpose the co-efficient $(c_1,c_2,\ldots c_n)$ of the objective function and the right side constants $(b_1,b_2,\ldots b_n)$
3. Change the inequalities from '≤' to '≥' sign.
4. Minimize the objective function instead of maximizing it.

Example Problems

Write the dual of the given problems

Example 1

Min $Z_x = 2x_2 + 5x_3$

Subject to

$x_1+x_2 \geq 2$

$2x_1+x_2+6x_3 \leq 6$

$x_1 - x_2 +3x_3 = 4$

$x_1, x_2 , x_3 \geq 0$

Solution

Primal

Max $Z_x' = -2x_2 – 5x_3$

Subject to

$-x_1-x_2 \leq -2$

$2x_1+x_2+6x_3 \leq 6$

$x_1 - x_2 +3x_3 \leq 4$

$-x_1 + x_2 -3x_3 \leq -4$

$x_1, x_2 , x_3 \geq 0$

Dual

Min $Z_w = -2w_1 + 6w_2 + 4w_3 – 4w_4$

Subject to

$-w_1 + 2w_2 +w_3 –w_4 \geq 0$

$-w_1 + w_2 - w_3 +w_4 \geq -2$

$6w_2 + 3w_3 –3w_4 \geq -5$

$w_{1,} w_{2,} w_{3,} w_4 \geq 0$

Example 2

Min $Z_x = 3x_1 - 2x_2 + 4x_3$

Subject to

$3x_1+5x_2+ 4x_3 \geq 7$

$6x_1+x_2+3x_3 \geq 4$

$7x_1 - 2x_2 -x_3 \geq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2 , x_3 \geq 0$

Solution

Primal

Max $Z_x' = -3x_1 + 2x_2 - 4x_3$

Subject to

$-3x_1 - 5x_2 - 4x_3 \leq -7$

$-6x_1 - x_2 - 3x_3 \leq -4$

$-7x_1 + 2x_2 + x_3 \leq - 10$

$-x_1 + 2x_2 - 5x_3 \leq - 3$

$-4x_1 - 7x_2 + 2x_3 \leq - 2$

$x_1, x_2 , x_3 \geq 0$

Dual

Min $Z_w = -7w_1 - 4w_2 - 10w_3 – 3w_4 -2w_5$

Subject to

$-3w_1 - 6w_2 - 7w_3 –w_4 – 4w_5 \geq -3$

$-5w_1 - w_2 + 2w_3 + 2w_4 – 7w_5 \geq 2$

$-4w_1 - 3w_2 + w_3 - 5w_4 + 2w_5 \geq -4$

$w_{1,} w_{2,} w_{3,} w_{4,} w_5 \geq 0$

Example 3

Max $Z = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$x_1, x_2 \geq 0$

Solution

Primal

Max $Z_x = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq -4$

$x_1, x_2 \geq 0$

Dual

Min $Z_w = 6w_1 - 6w_2 + 4w_3 - 4w_4$

Subject to

$4w_1 - 4w_2 + w_3 - w_4 \geq 2$

$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$

$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$

$w_1, w_2, w_3, w_4 \geq 0$

Example 4

Min $Z_x = x_1 + x_2 + x_3$

Subject to

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$,x_3 is unrestricted in sign

Solution

Primal

Max $Z' = -x_1 - x_2 - x_3' + x_3''$

Subject to

$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$

$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$

$x_1 - 2x_2 \leq 3$

$-2x_2 + x_3' - x_3'' \leq -4$

$x_1, x_2, x_3', x_3'' \geq 0$

Dual

Min $Z_w = 5w_1 - 5w_2 + 3w_3 - 4w_4$

Subject to

$w_1 - w_2 + w_3 \geq -1$

$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$

$4w_1 - 4w_2 + w_4 \geq -1$

$-4w_1 + 4w_2 - w_4 \geq 1$

$w_1, w_2, w_3, w_4, \geq 0$

Example 5

Max $Z = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 – x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Solution

Primal

Max $Z_x = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Dual

Min $Z_w = 10w_1 + 2w_2 + 6w_3$

Subject to

$w_1 + 2w_2 + 2w_3 \geq 1$

$w_1 - 2w_3 \geq -1$

$w_1 - w_2 + 3w_3 \geq 3$

$w_1, w_2, w_3 \geq 0$

3.7 Primal –Dual Relationship

Weak duality property

If x is any feasible solution to the primal problem and w is any feasible solution to the dual problem then $CX \leq b^T W$. i.e. $Z_X \leq Z_W$

Strong duality property

If x* is an optimal solution for the primal problem and w* is the optimal solution for the dual problem then $CX^* = b^T W^*$ i.e. $Z_X = Z_W$

Complementary optimal solutions property

At the final iteration, the simplex method simultaneously identifies an optimal solution x* for primal problem and a complementary optimal solution w* for the dual problem where $Z_X = Z_W$.

Symmetry property

For any primal problem and its dual problem, all relationships between them must be symmetric because dual of dual is primal.

Fundamental duality theorem

- If one problem has feasible solution and a bounded objective function (optimal solution) then the other problem has a finite optimal solution.
- If one problem has feasible solution and an unbounded optimal solution then the other problem has no feasible solution
- If one problem has no feasible solution then the other problem has either no feasible solution or an unbounded solution.

If kth constraint of primal is equality then the dual variable w_k is unrestricted in sign

If pth variable of primal is unrestricted in sign then pth constraint of dual is an equality.

Complementary basic solutions property

Each basic solution in the primal problem has a complementary basic solution in the dual problem where $Z_X = Z_W$.

Complementary slackness property

The variables in the primal basic solution and the complementary dual basic solution satisfy the complementary slackness relationship as shown in the table.

Primal variable	**Associated dual variable**
Decision variable (x_j)	$Z_j - C_j$ (surplus variable) j = 1, 2, ..n
Slack variable (S_i)	W_i (decision variable) i = 1, 2, .. n

Duality and Simplex Method

1. Solve the given primal problem using simplex method. Hence write the solution of its dual

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7$

$x_1 \geq 0, x_2 \geq 0$

Solution

Primal form

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7$

$x_1 \geq 0,\ x_2 \geq 0$

SLPP

Max $Z = 30x_1 + 23x_2 + 29x_3 + 0s_1 + 0s_2$

Subject to

$6x_1 + 5x_2 + 3x_3 + s_1 = 26$

$4x_1 + 2x_2 + 6x_3 + s_2 = 7$

$x_1, x_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	30	23	29	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min Ratio X_B / X_K
s_1	0	26	6	5	3	1	0	26/6
s_2	0	7	4	2	6	0	1	7/4→
			↑					
	Z = 0		-30	-23	-29	0	0	$\leftarrow\Delta_j$
s_1	0	31/2	0	2	-6	1	-3/2	31/4
x_1	30	7/4	1	1/2	3/2	0	1/4	7/2→
				↑				
	Z = 105/2		0	-8	16	0	15/2	$\leftarrow\Delta_j$
s_1	0	17/2	-4	0	-12	1	-5/2	
x_2	23	7/2	2	1	3	0	1/2	
	Z =161/2		16	0	40	0	23/2	$\leftarrow\Delta_j$

$\Delta_j \geq 0$ so the optimal solution is $Z = 161/2$, $x_1 = 0$, $x_2 = 7/2$, $x_3 = 0$.

The optimal solution to the dual of the above problem will be

$Z_w^* = 161/2$, $w_1 = \Delta_4 = 0$, $w_2 = \Delta_5 = 23/2$

In this way we can find the solution to the dual without actually solving it.

2. Use duality to solve the given problem. Also read the solution of its primal.

Min $Z = 3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max Z' = $-3x_1 - x_2$

Subject to

$- x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq - 2$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

Min $Z_w = -w_1 - 2w_2$

Subject to

$-w_1 - 2w_2 \geq -3$

$-w_1 - 3w_2 \geq -1$

$w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w' = w_1 + 2w_2 + 0s_1 + 0s_2$

Subject to

$w_1 + 2w_2 + s_1 = 3$

$w_1 + 3w_2 + s_2 = 1$

$w_{1,} w_2, s_1, s_2 \geq 0$

		C_j→	1	2	0	0	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	Min Ratio W_B / W_K
s_1	0	3	1	2	1	0	3/2
s_2	0	1	1	3	0	1	1/3←
	$Z_w' = 0$		-1	↑ -2	0	0	←Δ_j
s_1	0	7/3	1/3	0	1	-2/3	7
w_2	2	1/3	1/3	1	0	1/3	1→
	$Z_w' = 2/3$		↑ -1/3	0	0	2/3	←Δ_j
s_1	0	2	0	-1	1	-1	

w_1	1	1	1	3	0	1	
		$Z_w' = 1$	0	1	0	1	$\leftarrow \Delta_j$

$\Delta_j \geq 0$ so the optimal solution is $Z_w' = 1$, $w_1 = 1$, $w_2 = 0$

The optimal solution to the primal of the above problem will be

$Z_x^* = 1$, $x_1 = \Delta_3 = 0$, $x_2 = \Delta_4 = 1$

3. Write down the dual of the problem and solve it.

$\text{Max } Z = 4x_1 + 2x_2$

Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$

$x_1 \geq 0, x_2 \geq 0$

Solution

Primal

$\text{Max } Z = 4x_1 + 2x_2$

Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$

$x_1 \geq 0, x_2 \geq 0$

Dual

$\text{Min } Z_w = -3w_1 - 2w_2$

Subject to

$-w_1 - w_2 \geq 4$

$-w_1 + w_2 \geq 2$

$w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

$\text{Max } Z_w' = 3w_1 + 2w_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

Subject to

$-w_1 - w_2 - s_1 + a_1 = 4$

$-w_1 + w_2 - s_2 + a_2 = 2$

$w_1, w_2, s_1, s_2, a_1, a_2 \geq 0$

	$C_j \rightarrow$		3	2	0	0	-M	-M	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	A_2	Min Ratio W_B / W_K
a_1	-M	4	-1	-1	-1	0	1	0	-
a_2	-M	2	-1	1	0	-1	0	1	2→
				↑					
	$Z_w' = -6M$		2M - 3	-2	M	M	0	0	←Δ_j
a_1	-M	6	-2	0	-1	-1	1	X	
w_2	2	2	-1	1	0	-1	0	X	
	$Z_w' = -6M+4$		2M-5	0	M	M-2	0	X	←Δ_j
a_1	-M	6	-2	0	-1	-1	1	X	
w_2	2	2	-1	1	0	-1	0	X	
	$Z_w' =$ -6M+4		2M-5	0	M	M-2	0	X	←Δ_j

$\Delta_j \geq 0$ and at the positive level an artificial vector (a_1) appears in the basis. Therefore the dual problem does not posses any optimal solution. Consequently there exists no finite optimum solution to the given problem.

4. Use duality to solve the given problem.

Min $Z = x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max $Z' = -x_1 + x_2$

Subject to

$- 2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

Min $Z_w = -2w_1 - w_2$

Subject to

$-2w_1 + w_2 \geq -1$

$-w_1 + w_2 \geq 1$

$w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w{}' = 2w_1 + w_2 + 0s_1 + 0s_2 - Ma_1$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_{1,} w_2, s_1, s_2 \geq 0$

Auxiliary LPP

Max $Z_w{}' = 0w_1 + 0w_2 + 0s_1 + 0s_2 - 1a_1$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_{1,} w_2, s_1, s_{2,} a_1 \geq 0$

Phase I

	$C_j\rightarrow$		0	0	0	0	-1	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	1	2	-1	1	0	0	-
a_1	-1	1	-1	1	0	-1	1	1→
				↑				
	$Z_w{}' = -1$		1	-1	0	1	0	$\leftarrow\Delta_j$
s_1	0	2	1	0	1	-1	X	
w_2	0	1	-1	1	0	-1	X	
	$Z_w{}' = 0$		0	0	0	0	X	$\leftarrow\Delta_j$

$\Delta_j \geq 0$ and no artificial vector appear at the positive level of the basis. Hence proceed to phase II

Phase II

		$C_j \rightarrow$	2	1	0	0	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	2	1	0	1	-1	2→
w_2	1	1	-1	1	0	-1	-
			↑				
	$Z_w{}' = 1$		-3	0	0	-1	←Δ_j
w_1	2	2	1	0	1	-1	-
w_2	1	3	0	1	1	-2	-
						↑	
	$Z_w{}' = 7$		0	0	3	-4	←Δ_j

$\Delta_j = -4$ and all the elements of s_2 are negative; hence we cannot find the outgoing vector. This indicates there is an unbounded solution. Consequently by duality theorem the original primal problem will have no feasible solution.

5. Solve the given primal problem using simplex method. Hence write the solution of its dual

Max $Z = 40x_1 + 50x_2$
Subject to
$2x_1 + 3x_2 \leq 3$
$8x_1 + 4x_2 \leq 5$
$x_1 \geq 0, x_2 \geq 0$

Solution

Primal form

Max $Z = 40x_1 + 50x_2$
Subject to
$2x_1 + 3x_2 \leq 3$
$8x_1 + 4x_2 \leq 5$
$x_1 \geq 0, x_2 \geq 0$

SLPP

Max $Z_x = 40x_1 + 50x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 + 3x_2 + s_1 = 3$

$8x_1 + 4x_2 + s_2 = 5$

$x_1, x_2, s_1, s_2 \geq 0$

	C_j →		40	50	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	3	2	3	1	0	1→
s_2	0	5	8	4	0	1	5/4
	$Z_x = 0$		-40	↑ -50	0	0	←Δ_j
x_2	50	1	2/3	1	1/3	0	3/2
s_2	0	1	16/3	0	-4/3	1	3/16→
	$Z_x = 50$		↑ -20/3	0	50/3	0	←Δ_j
x_2	50	7/8	0	1	1/2	-1/8	
x_1	40	3/16	1	0	-1/4	3/16	
	$Z_x = 205/4$		0	0	15	5/4	←Δ_j

$\Delta_j \geq 0$ so the optimal solution is $Z = 205/4$, $x_1 = 3/16$, $x_2 = 7/8$

The optimal solution to the dual of the above problem will be

$Z_w^* = 205/4$, $w_1 = \Delta_4 = 15$, $w_2 = \Delta_5 = 5/4$

Computational Procedure of Dual Simplex Method

The iterative procedure is as follows

Step 1 - First convert the minimization LPP into maximization form, if it is given in the minimization form.

Step 2 - Convert the '≥' type inequalities of given LPP, if any, into those of '≤' type by multiplying the corresponding constraints by -1.

Step 3 – Introduce slack variables in the constraints of the given problem and obtain an initial basic solution.

Step 4 – Test the nature of Δ_j in the starting table

- If all Δ_j and X_B are non-negative, then an optimum basic feasible solution has been attained.
- If all Δ_j are non-negative and at least one basic variable X_B is negative, then go to step 5.
- If at least Δ_j one is negative, the method is not appropriate.

Step 5 – Select the most negative X_B. The corresponding basis vector then leaves the basis set B. Let X_r be the most negative basic variable.

Step 6 – Test the nature of X_r

- If all X_r are non-negative, then there does not exist any feasible solution to the given problem.
- If at least one X_r is negative, then compute Max (Δ_j / X_r) and determine the least negative for incoming vector.

Step 7 – Test the new iterated dual simplex table for optimality.

Repeat the entire procedure until either an optimum feasible solution has been attained in a finite number of steps.

Worked Examples

Example 1

Minimize $Z = 2x_1 + x_2$

Subject to

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Rewrite the given problem in the form

Maximize $Z2 - = {}' x_1 - x_2$

Subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 \quad - 2x_2 \leq -3$

$x_1, x_2 \geq 0$

Step 2 – Adding slack variables to each constraint

Maximize $Z2 - = {}' x_1 - x_2$

Subject to

$-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_2 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-3	-3	-1	1	0	0	
s_2	0	-6	-4	-3	0	1	0	→ outgoing
s_3	0	-3	-1	-2	0	0	1	
				↑				
	Z0 = ′		2	1	0	0	0	$\leftarrow \Delta_j$

Step 4 – To find the leaving vector

Min (-3, -6, -3) = -6. Hence s_2 is outgoing vector

Step 5 – To find the incoming vector

Max (Δ_1 / x_{21}, Δ_2 / x_{22}) = (2/-4, 1/-3) = -1/3. So x_2 is incoming vector

Step 6 –The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-1	-5/3	0	1	-1/3	0	→ outgoing
x_2	-1	2	4/3	1	0	-1/3	0	
s_3	0	1	5/3	0	0	-2/3	1	
			↑					
	Z2- = ′		2/3	0	0	1/3	0	$\leftarrow \Delta_j$

Step 7 – To find the leaving vector

Min (-1, 2, 1) = -1. Hence s_1 is outgoing vector

Step 8 – To find the incoming vector

Max (Δ_1 / x_{11}, Δ_4 / x_{14}) = (-2/5, -1) = -2/5. So x_1 is incoming vector

Step 9 –The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x1	-2	3/5	1	0	-3/5	1/5	0	
x2	-1	6/5	0	1	4/5	-3/5	0	
s3	0	0	0	0	1	-1	1	

	Z- = ′ 12/5	0	0	2/5	1/5	0	←Δ_j

Step 10 – $\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max Z12/5- = ′, Z = 12/5, and x_1=3/5, x_2 = 6/5

Example 2

Minimize Z = $3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Maximize Z3 – = ′$x_1 - x_2$

Subject to

$-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

$x_1, x_2 \geq 0$

SLPP

Maximize Z3 – = ′$x_1 - x_2$

Subject to

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	
s_1	0	-1	-1	-1	1	0	
s_2	0	-2	-2	-3	0	1	→
	Z0 = ′		3	↑ 1	0	0	←Δ_j
s_1	0	-1/3	-1/3	0	1	-1/3	→
x_2	-1	2/3	2/3	1	0	-1/3	

	Z2/3- = ′		7/3	0	0	↑ 1/3	←Δ_j
s_2	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	Z1- = ′		2	0	1	0	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max Z1- = ′, Z = 1, and $x_1 = 0$, $x_2 = 1$

Example 3

Max $Z = -2x_1 - x_3$

Subject to

$x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Solution

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 \leq -5$

$-x_1 + 2x_2 - 4x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

SLPP

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 + s_1 = -5$

$-x_1 + 2x_2 - 4x_3 + s_2 = -8$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-2	0	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	
s_1	0	-5	-1	-1	1	1	0	
s_2	0	-8	-1	2	-4	0	1	→
	Z = 0		2	0	↑ 1	0	0	←Δ_j
s_1	0	-7	-5/4	-1/2	0	1	1/4	→
x_3	-1	2	1/4	-1/2	1	0	-1/4	

	Z = -2		7/4	↑ 1/2	0	0	1/4	←Δ_j
x_2	0	14	5/2	1	0	-2	-1/2	
x_3	-1	9	3/2	0	1	-1	-1/2	
	Z = -9		1/2	0	0	1	1/2	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Z = -9, and $x_1 = 0$, $x_2 = 14$, $x_3 = 9$

Introduction to Transportation Problem

The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

It can also be defined as to ship goods from various origins to various destinations in such a manner that the transportation cost is a minimum.

The availability as well as the requirements is finite. It is assumed that the cost of shipping is linear.

Mathematical Formulation

Let there be m origins, i^{th} origin possessing a_i units of a certain product

Let there be n destinations, with destination j requiring b_j units of a certain product

Let c_{ij} be the cost of shipping one unit from i^{th} source to j^{th} destination

Let x_{ij} be the amount to be shipped from i^{th} source to j^{th} destination

It is assumed that the total availabilities Σa_i satisfy the total requirements Σb_j i.e.

$\Sigma a_i = \Sigma b_j$ (i = 1, 2, 3 … m and j = 1, 2, 3 …n)

The problem now, is to determine non-negative of x_{ij} satisfying both the availability constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, .., m$$

as well as requirement constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{for } j = 1, 2, .., n$$

and the minimizing cost of transportation (shipping)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} \quad \text{(objective function)}$$

This special type of LPP is called as **Transportation Problem**.

Tabular Representation

Let ‘m’ denote number of factories (F_1, F_2 … F_m)

Let ‘n’ denote number of warehouse (W_1, W_2 … W_n)

W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	c_{11}	c_{12}	…	c_{1n}	a_1
F_2	c_{21}	c_{22}	…	c_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	c_{m1}	c_{m2}	…	c_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$

W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	x_{11}	x_{12}	…	x_{1n}	a_1
F_2	x_{21}	x_{22}	…	x_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
Fm	xm1	xm2	…	xmn	am
Required	b1	b2	…	bn	Σai = Σbj

In general these two tables are combined by inserting each unit cost c_{ij} with the corresponding amount x_{ij} in the cell (i, j). The product c_{ij} x_{ij} gives the net cost of shipping units from the factory F_i to warehouse W_j.

Some Basic Definitions

- **Feasible Solution**

A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called as feasible solution.

- **Basic Feasible Solution**

  A feasible solution to ‘m’ origin, ‘n’ destination problem is said to be basic if the number of positive allocations are m+n-1. If the number of allocations is less than m+n-1 then it is called as **Degenerate Basic Feasible Solution**. Otherwise it is called as Non-Degenerate Basic Feasible Solution.

- **Optimum Solution**

  A feasible solution is said to be optimal if it minimizes the total transportation cost.

North-West Corner Rule

Step 1

- The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the table.
- The maximum possible amount is allocated here i.e. $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1,1) of the transportation table.

Step 2

i. If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

ii. If $b_1 < a_1$, move horizontally right side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

iii. If $b_1 = a_1$, there is tie for the second allocation. One can make a second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2)$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1)$ in the cell (2, 1)

Step 3

Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

Find the initial basic feasible solution by using North-West Corner Rule

W→ F	W_1 W_2 W_3 W_4	Factory

↓					Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

1.

Solution

	W_1	W_2	W_3	W_5	Availability
F_1	5 (19)	2 (30)			7 2 0
F_2		6 (30)	3 (40)		9 3 0
F_3			4 (70)	14 (20)	18 14 0
Requirement	5 0	8 6 0	7 4 0	14 0	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

	D_1	D_2	D_3	D_4	Supply
O_1	1	5	3	3	34
O_2	3	3	1	2	15

O$_3$	0	2	2	3	12
O$_4$	2	7	2	4	19
Demand	21	25	17	17	80

Solution

	D$_1$	D$_2$	D$_3$	D$_4$	Supply
O$_1$	21 (1)	13 (5)			34 13 0
O$_2$		12 (3)	3 (1)		15 3 0
O$_3$			12 (2)		12 0
O$_4$			2 (2)	17 (4)	19 17
Demand	21	25	17	17	
	0	12	14	0	
		0	2		
			0		

Initial Basic Feasible Solution

$x_{11} = 21$, $x_{12} = 13$, $x_{22} = 12$, $x_{23} = 3$, $x_{33} = 12$, $x_{43} = 2$, $x_{44} = 17$

The transportation cost is 21 (1) + 13 (5) + 12 (3) + 3 (1) + 12 (2) + 2 (2) + 17 (4) = Rs. 221

3.

From	To					Supply
	2	11	10	3	7	4
	1	4	7	2	1	8
	3	1	4	8	12	9
Demand	3	3	4	5	6	

Solution

From	To					Supply
	3 (2)	1 (11)				4 1 0
		2 (4)	4 (7)	2 (2)		8 6 2 0
				3 (8)	6 (12)	9 6 0
	3	3	4	5	6	
Demand	0	2	0	3	0	
		0		0		

Initial Basic Feasible Solution

$x_{11} = 3, x_{12} = 1, x_{22} = 2, x_{23} = 4, x_{24} = 2, x_{34} = 3, x_{35} = 6$

The transportation cost is 3 (2) + 1 (11) + 2 (4) + 4 (7) + 2 (2) + 3 (8) + 6 (12) = Rs. 153

Lowest Cost Entry Method (Matrix Minima Method)

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Allocate $x_{ij} = \min (a_i, b_j)$ in the cell (i, j)

Step 2

- If $x_{ij} = a_i$, cross out the i^{th} row of the table and decrease b_j by a_i. Go to step 3.
- If $x_{ij} = b_j$, cross out the j^{th} column of the table and decrease a_i by b_j. Go to step 3.
- If $x_{ij} = a_i = b_j$, cross out the i^{th} row or j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Find the initial basic feasible solution using Matrix Minima method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	7 (20)	3
	5	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9

F_3	3 (40)	8 (8)	(70)	7 (20)	X
	2	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	2 (70)	(30)	7 (40)	(60)	X
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$

The transportation cost is 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20) = Rs. 814

2.

	To					Availability
From	2	11	10	3	7	4
	1	4	7	2	1	8
	3	9	4	8	12	9
Requirement	3	3	4	5	6	

Solution

To

From				4 (3)		4 0
	3 (1)				5 (1)	8 5 0
		3 (9)	4 (4)	1 (8)	1 (12)	9 5 4 1 0
	3	3	4	5	6	
	0	0	0	1	1	
				0	0	

Initial Basic Feasible Solution

$x_{14} = 4$, $x_{21} = 3$, $x_{25} = 5$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 1$, $x_{35} = 1$

The transportation cost is 4 (3) + 3 (1) + 5(1) + 3 (9) + 4 (4) + 1 (8) + 1 (12) = Rs. 78

Vogel's Approximation Method (Unit Cost Penalty Method)

Step1

For each row of the table, identify the **smallest** and the **next to smallest cost**. Determine the difference between them for each row. These are called **penalties**. Put them aside by enclosing them in the parenthesis against the respective rows. Similarly compute penalties for each column.

Step 2

Identify the row or column with the largest penalty. If a tie occurs then use an arbitrary choice. Let the largest penalty corresponding to the ith row have the cost c_{ij}. Allocate the largest possible amount x_{ij} = min (a_i, b_j) in the cell (i, j) and cross out either ith row or jth column in the usual manner.

Step 3

Again compute the row and column penalties for the reduced table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Find the initial basic feasible solution using vogel's approximation method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	19	30	50	10	7	19-10=9
F_2	70	30	40	60	9	40-30=10
F_3	40	8	70	20	18	20-8=12
Requirement	5	8	7	14		
Penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	(19)	(30)	(50)	(10)	7	9
F_2	(70)	(30)	(40)	(60)	9	10
F_3	(40)	**8**(8)	(70)	(20)	18/10	12
Requirement	5	8/0	7	14		
Penalty	21	22	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	9
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	(20)	18/10	20
Requirement	5/0	X	7	14		
Penalty	21	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	18/10/0	50
Requirement	X	X	7	14/4		
Penalty	X	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	7/2/0	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	7	14/4/2		
Penalty	X	X	10	50		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X

F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

The transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2.

		Stores				Availability
		I	II	III	IV	
	A	21	16	15	13	11
Warehouse	B	17	18	14	23	13
	C	32	27	18	41	19
Requirement		6	10	12	15	

Solution

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	(13)	11	2
Warehouse	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15		
Penalty		4	2	1	10		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	11/0	2
Warehouse	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4		
Penalty		4	2	1	10		

		Stores				Availability	Penalty
		I	II	III	IV		
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	(17)	(18)	(14)	**4**(23)	13/9	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4/0		
Penalty		15	9	4	18		

		Stores				Availability	Penalty
		I	II	III	IV		
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	**6**(17)	(18)	(14)	**4**(23)	13/9/3	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6/0	10	12	X		
Penalty		15	9	4	X		

		Stores				Availability	Penalty
		I	II	III	IV		
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	**6**(17)	**3**(18)	(14)	**4**(23)	13/9/3/0	4
	C	(32)	(27)	(18)	(41)	19	9
Requirement		X	10/7	12	X		
Penalty		X	9	4	X		

		Stores				Availability	Penalty
		I	II	III	IV		
Warehouse	A	(21)	(16)	(15)	**11**(13)	X	X
	B	**6**(17)	**3**(18)	(14)	**4**(23)	X	X
	C	(32)	**7**(27)	**12**(18)	(41)	X	X
Requirement		X	X	X	X		
Penalty		X	X	X	X		

Initial Basic Feasible Solution

$x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$

The transportation cost is 11 (13) + 6 (17) + 3 (18) + 4 (23) + 7 (27) + 12 (18) = Rs. 796

Transportation Algorithm for Minimization Problem (MODI Method)

Step 1

Construct the transportation table entering the origin capacities a_i, the destination requirement b_j and the cost c_{ij}

Step 2

Find an initial basic feasible solution by vogel's method or by any of the given method.

Step 3

For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$, calculate the values of u_i and v_j on the transportation table

Step 4

Compute the cost differences $d_{ij} = c_{ij} – (u_i + v_j)$ for all the non-basic cells

Step 5

Apply optimality test by examining the sign of each d_{ij}

- If all $d_{ij} \geq 0$, the current basic feasible solution is optimal
- If at least one $d_{ij} < 0$, select the variable x_{rs} (most negative) to enter the basis.
- Solution under test is not optimal if any d_{ij} is negative and further improvement is required by repeating the above process.

Step 6

Let the variable x_{rs} enter the basis. Allocate an unknown quantity ϴ to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount ϴ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7

Assign the largest possible value to the ϴ in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step 8

Now, return to step 3 and repeat the process until an optimal solution is obtained.

3.3 Worked Examples

Example 1

Find an optimal solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

1. **Applying vogel's approximation method for finding the initial basic feasible solution**

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X

F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Minimum transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2. **Check for Non-degeneracy**

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

3. **Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$**

					u_i
	• (19)			• (10)	u_1= -10
			• (40)	• (60)	u_2 = 40
		• (8)		• (20)	u_3 = 0
v_j	v_1 = 29	v_2 = 8	v_3 = 0	v_4 = 20	

Assign a 'u' value to zero. (Convenient rule is to select the u_i, which has the largest number of allocations in its row)

Let $u_3 = 0$, then

$u_3 + v_4$= 20 which implies $0 + v_4 = 20$, so $v_4 = 20$

$u_2 + v_4$= 60 which implies $u_2 + 20 = 60$, so $u_2 = 40$

$u_1 + v_4$= 10 which implies $u_1 + 20 = 10$, so $u_1 = -10$

$u_2 + v_3$= 40 which implies $40 + v_3 = 40$, so $v_3 = 0$

$u_3 + v_2$= 8 which implies $0 + v_2 = 8$, so $v_2 = 8$

$u_1 + v_1$= 19 which implies $-10 + v_1$= 19, so $v_1 = 29$

4. **Calculation of cost differences for non basic cells $d_{ij} = c_{ij} – (u_i + v_j)$**

c_{ij}

●	(30)	(50)	●
(70)	(30)	●	●
(40)	●	(70)	●

$u_i + v_j$

●	-2	-10	●
69	48	●	●
29	●	0	●

$d_{ij} = c_{ij} – (u_i + v_j)$

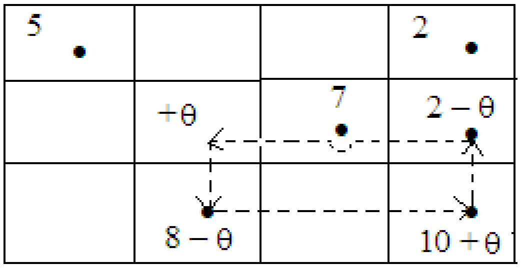
●	32	60	●
1	**-18**	●	●
11	●	70	●

5. **Optimality test**

$d_{ij} < 0$ i.e. $d_{22} = -18$

so x_{22} is entering the basis

6. **Construction of loop and allocation of unknown quantity Θ**



We allocate Θ to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Θ in the marked cell. The value of Θ is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Θ, 2-Θ) = 0 which gives Θ = 2. Therefore x_{24} is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

7. **Improved Solution**

					u_i
	• (19)			• (10)	$u_1 = -10$
		• (30)	• (40)		$u_2 = 22$
		• (8)		• (20)	$u_3 = 0$
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	

c_{ij}

•	(30)	(50)	•
(70)	•	•	(60)
(40)	•	(70)	•

$u_i + v_j$

•	-2	8	•
51	•	•	42
29	•	18	•

$d_{ij} = c_{ij} – (u_i + v_j)$

•	32	42	•
19	•	•	18
11	•	52	•

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.743

Example 2

Solve by lowest cost entry method and obtain an optimal solution for the following problem

				Available
	50	30	220	1
From	90	45	170	3
	250	200	50	4
Required	4	2	2	

Solution

By lowest cost entry method

				Available
		1(30)		1/0
From	2(90)	1(45)		3/2/0
	2(250)		2(50)	4/2/0
Required	4/2/2	2/1/0	2/0	

Minimum transportation cost is 1 (30) + 2 (90) + 1 (45) + 2 (250) + 2 (50) = Rs. 855

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 3 – 1 = 5 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		● (30)		u_1= -15
	● (90)	● (45)		u_2 = 0
	● (250)		● (50)	u_3 = 160
v_j	v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

50	●	220

$u_i + v_j$

75	●	-125

●	●	170
●	200	●

●	●	-110
●	205	●

$d_{ij} = c_{ij} – (u_i + v_j)$

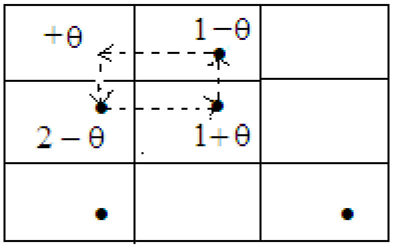
-25	●	345
●	●	280
●	-5	●

Optimality test

$d_{ij} < 0$ i.e. $d_{11} = -25$ is most negative

So x_{11} is entering the basis

Construction of loop and allocation of unknown quantity Ө



min(2-Ө, 1-Ө) = 0 which gives Ө = 1. Therefore x_{12} is outgoing as it becomes zero.

1(50)		
1(90)	2(45)	
2(250)		2(50)

Minimum transportation cost is 1 (50) + 1 (90) + 2 (45) + 2 (250) + 2 (50) = Rs. 830

II Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	● (50)			u_1= -40
	● (90)	● (45)		u_2 = 0
	● (250)		● (50)	u_3 = 160
v_j	v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

●	30	220
●	●	170
●	200	●

$u_i + v_j$

●	5	-150
●	●	-110
●	205	●

$d_{ij} = c_{ij} – (u_i + v_j)$

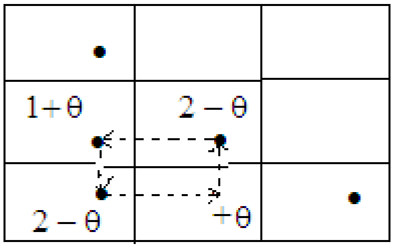
●	25	370
●	●	280
●	-5	●

Optimality test

$d_{ij} < 0$ i.e. $d_{32} = -5$

So x_{32} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



$2 – \Theta = 0$ which gives $\Theta = 2$. Therefore x_{22} and x_{31} is outgoing as it becomes zero.

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	• (50)			$u_1 = -40$
	• (90)	• (45)		$u_2 = 0$
		• (200)	• (50)	$u_3 = 155$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -105$	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

•	30	220
•	•	170
250	•	•

$u_i + v_j$

•	5	-145
•	•	-105
245	•	•

$d_{ij} = c_{ij} – (u_i + v_j)$

•	25	365
•	•	275
5	•	•

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.820

Introduction to Assignment Problem

In assignment problems, the objective is to assign a number of jobs to the equal number of persons at a minimum cost of maximum profit.

Suppose there are ‘n’ jobs to be performed and ‘n’ persons are available for doing these jobs. Assume each person can do each job at a time with a varying degree of efficiency. Let c_{ij} be the cost of ith person assigned to jth job. Then the problem is to find an assignment so that the total cost for performing all jobs is minimum. Such problems are known as **assignment problems.**

These problems may consist of assigning men to offices, classes to the rooms or problems to the research team etc.

Mathematical formulation

Cost matrix: $c_{ij} = \begin{matrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ . & & & & \\ . & & & & \\ . & & & & \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{matrix}$

$$\text{Minimize cost : } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad i = 1, 2, .., n \;\; j = 1, 2, .., n$$

Subject to restrictions of the form

$$x_{ij} = \begin{cases} 1 \text{ if ith person is assigned jth job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{(one job is done by the ith person, } i = 1,2,\dots n\text{)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{(only one person should be assigned the jth job, } j = 1,2,\dots n\text{)}$$

Where x_{ij} denotes that jth job is to be assigned to the ith person.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

1.2 Algorithm for Assignment Problem (Hungarian Method)

Step 1

Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows (Row reduced matrix).

Step 2

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus first modified matrix is obtained.

Step 3

Draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities

- If N = n, the number of rows (columns) of the given matrix then an optimal assignment can be made. So make the zero assignment to get the required solution.
- If N < n then proceed to step 4

Step 4

Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 5

Repeat step 3 and step 4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N = n.

Step 6

To make zero assignment - examine the rows successively until a row-wise exactly single zero is found; mark this zero by ‘ ‘to make the assignment. Then, mark a ‘X’ over all zeroes if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Step 7

Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If there lies more than one of the unmarked zeroes in any column or row, then mark ‘ ‘one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8

Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment.

1.3 Worked Examples

Example 1

A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

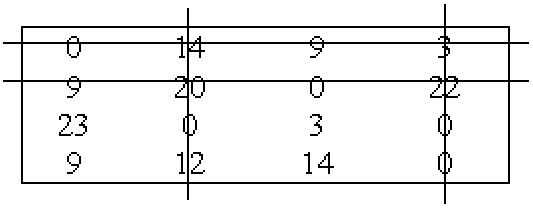
		Subordinates			
		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution

Row Reduced Matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

I Modified Matrix



0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

N = 4, n = 4

Since N = n, we move on to zero assignment

Zero assignment

$$\begin{bmatrix} \boxed{0} & 14 & 9 & 3 \\ 9 & 20 & \boxed{0} & 22 \\ 23 & \boxed{0} & 3 & \cancel{0} \\ 9 & 12 & 14 & \boxed{0} \end{bmatrix}$$

Optimal assignment	A – I	B – III	C – II	D – IV
Man-hours	8	4	19	10

Total man-hours = 8 + 4 + 19 + 10 = 41 hours

Example 2

A car hire company has one car at each of five depots a, b, c, d and e. a customer requires a car in each town namely A, B, C, D and E. Distance (kms) between depots (origins) and towns (destinations) are given in the following distance matrix

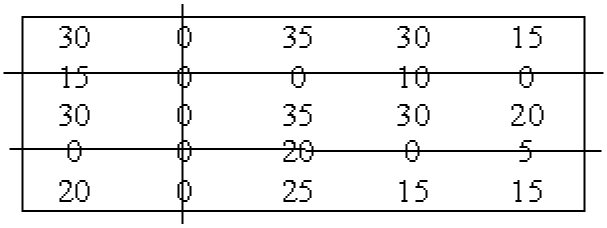
	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Solution

Row Reduced Matrix

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

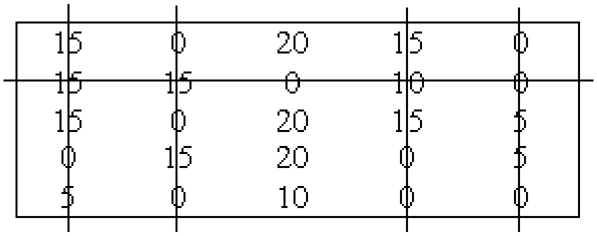
I Modified Matrix



30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

N < n i.e. 3 < 5, so move to next modified matrix

II Modified Matrix

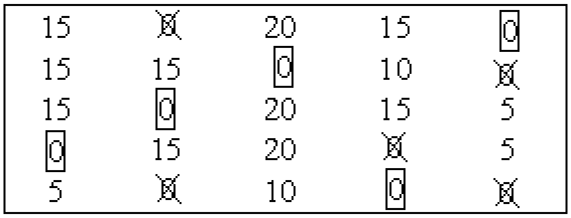


15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

N = 5, n = 5

Since N = n, we move on to zero assignment

Zero assignment



15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Route	A – e	B – c	C – b	D – a	E – d
Distance	200	130	110	50	80

Minimum distance travelled = 200 + 130 + 110 + 50 + 80 = 570 kms

Example 3

Solve the assignment problem whose effectiveness matrix is given in the table

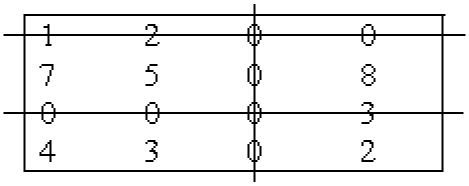
	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution

Row-Reduced Matrix

4	15	0	16
10	18	0	24
3	13	0	19
7	16	0	18

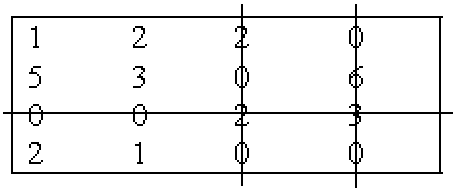
I Modified Matrix



1	2	0	0
7	5	0	8
0	0	0	3
4	3	0	2

N < n i.e 3 < 4, so II modified matrix

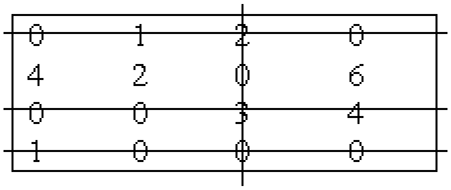
II Modified Matrix



1	2	2	0
5	3	0	6
0	0	2	3
2	1	0	0

N < n i.e 3 < 4

III Modified matrix



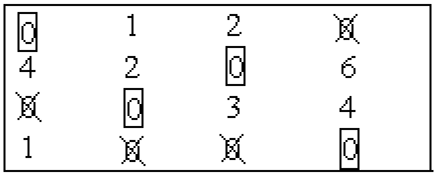
0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Since N = n, we move on to zero assignment

Zero assignment

Multiple optimal assignments exists

Solution - I



$\boxed{0}$	1	2	$\xcancel{0}$
4	2	$\boxed{0}$	6
$\xcancel{0}$	$\boxed{0}$	3	4
1	$\xcancel{0}$	$\xcancel{0}$	$\boxed{0}$

Optimal assignment	A – 1	B – 3	C – 2	D - 4
Value	49	45	62	66

Total cost = 49 + 45 + 62 + 66 = 222 units

Solution – II

$\xcancel{0}$	1	2	$\boxed{0}$
4	2	$\boxed{0}$	6
$\boxed{0}$	$\xcancel{0}$	3	4
1	$\boxed{0}$	$\xcancel{0}$	$\xcancel{0}$

Optimal assignment	A – 4	B – 3	C – 1	D – 2
Value	61	45	52	64

Minimum cost = 61 + 45 + 52 + 64 = 222 units

Example 4

Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table. Assuming that each mechanic can be assigned only one job, determine the minimum time assignment.

	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Solution

Row Reduced Matrix

2	0	4	3	6
2	5	0	4	3
4	1	0	2	5
4	0	3	6	2
0	2	3	1	7

I Modified Matrix

2	0	4	2	4
2	5	0	3	1
4	1	0	1	3
4	0	3	5	0
0	2	3	0	5

N < n

II Modified Matrix

1	0	4	1	3
1	5	0	2	0
3	1	0	0	2
4	1	4	5	0
0	3	4	0	5

N = n

Zero assignment

1	$\boxed{0}$	4	1	3
1	5	$\boxed{0}$	2	$\xcancel{0}$
3	1	$\xcancel{0}$	$\boxed{0}$	2
4	1	4	5	$\boxed{0}$
$\boxed{0}$	3	4	$\xcancel{0}$	5

Optimal assignment	M1 – J2	M2 – J3	M3 – J4	M4 – J5	M5 – J1
Hours	5	7	6	5	4

Minimum time = 5 + 7 + 6 + 5 + 4 = 27 hours

5 Maximal Assignment Problem

Example 1

A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution

Subtract all the elements from the highest element

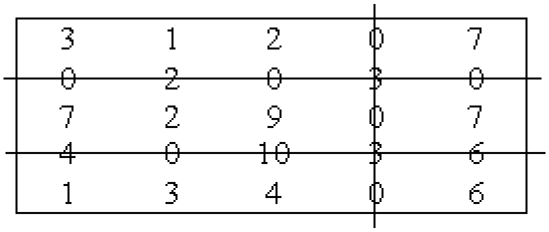
Highest element = 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row Reduced matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

I Modified Matrix



3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

N < n i.e. 3 < 5

II Modified Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	4	6
0	2	3	0	5

N < n i.e. 4 < 5

III Modified Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

N = n

Zero assignment

1	$\xcancel{0}$	$\boxed{0}$	$\xcancel{0}$	5
$\xcancel{0}$	3	$\xcancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

Optimal assignment 1 – C 2 – E 3 – D 4 – B 5 – A

Maximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50