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UNIT III FINITE ELEMENT METHOD - FEM

Introduction

a. It is a numerical technique.

b. Developed for air craft industry.

c. Used for solid mechanics, fluid flow analysis, Heat transfer analysis, electric & magnetic field analysis, & in civil Engineering for

d. analysis of beams, space frames, plates, shells, folded plates, rock mechanics problems, foundation analysis, seepage analysis etc.

d. This method is used mostly for design of ships, aircrafts, space crafts, electric motors & heat engines.

Terminology:

Continuum : A domain in which matter exists at every point (or) infinite number of connected particles is continuum.

Finite elements: The element obtained by the process of discretization of a domain is finite element.

Domain: In civil Engineering - a domain is a system or structure.

Discretization: A process of sub dividing the given domain suitably for analysing & designing is discretization.

Nodes : Nodes are the points at which different elements are assembled to obtain domain system.

Linear elements : Finite element with straight sides.

Higher order elements: Finite element with curved sides.

Procedure for FEM

Step I : → Discretise the continuum, - select suitable length for 1D elements
→ Select Geometry (shape) of the elements for 2D & 3D ie △, ▭, □, ▱ etc.

Step II : Select Element Displacement functions:
Select proper field variables ie displacement for a structure. Stress & strains are functions of displacements.

Step III : Calculate element properties (k)

It uses element equilibrium equations to calculate element stiffness matrix (k).

$$\delta = k\,P$$

Step IV : obtain element Load vector [P]

It is obtained by loading on the element. The load can be point, vdl or uvl.

Step V : Assemble element properties - K

$$K = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$

Step VI : Impose boundary conditions & obtain structure displacement vector U
eg. for fixed support displacement is zero.

Step VII : $F = K\,U \quad \therefore\ U = [K]^{-1}\{F\}$

F = Known forces at co-ordinates

(2)

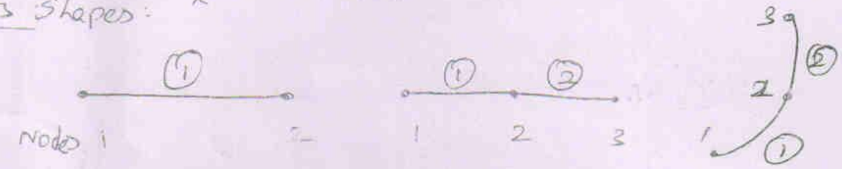
Step VII From structure displacement vector we can obtain displacement(s) at all co-ordinates.

Step VIII Assemble all element models to obtain domain. & hence the result.

→ Teach Discretisation available on next page

Elements shapes:

1D –

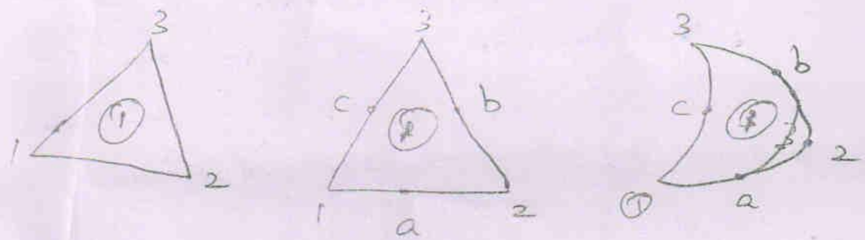


① – element

1, 2, ... – nodes

2D

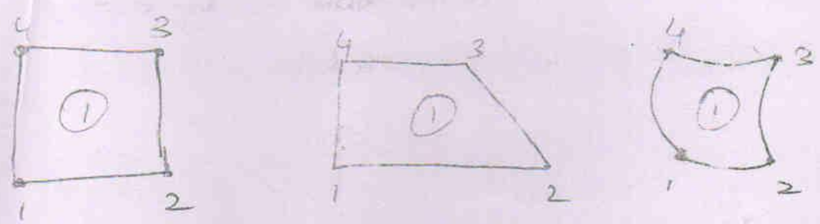
I



Triangular element

eg,

II

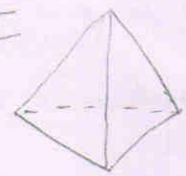


1, 2, 3 – Primary unknowns

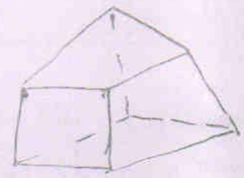
a, b, c – Secondary unknowns

3D Elements

I



Tetrahedron
(4)



Hexahedron
(6)

II Axisymmetric

III

Cone

Curved beams

Displacement functions: NN

$$[k][\delta] = F$$

K = stiffness matrix

δ = displacement vector (deflection or rotation)

F = Force vector

Advantages of FEM

1. It can handle structures with anisotropic (nonhomogeneous) property elements without difficulty
2. It can handle any shape, any loadings & any boundary conditions with ease & without making drastic assumptions.
3. It can handle non linearities of material & geometry

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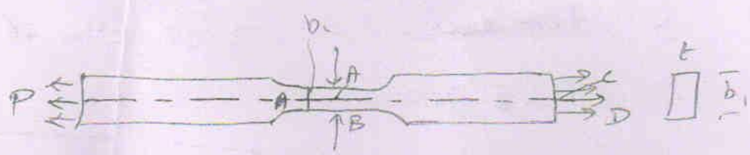
4. solution can be obtained for all problems
5. This method is useful for wide range of engineering applications

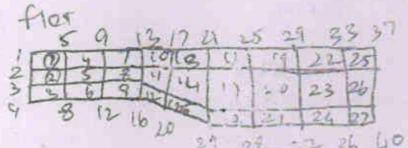
Disadvantages of FEM:

1. It provides approximate solutions.
2. Improper handling of physical like discretisation, boundary conditions leads to erroneous result.
3. The stiffness matrix may become singular & solution becomes impossible. like
    a. badly configured
    b. vast variations in elements sizes
    c. Improper choice of units
4. Proper cross checks is a must necessary, otherwise it may lead to gross errors.

Discretisation – Tension flat

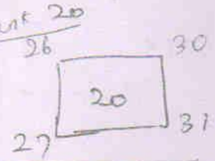


Quarter flat



elements – 27
nodes – 40

Element 20



Nodes 26, 27, 30, 31

Total displacement = nodes × 2 = 40×2 = 80

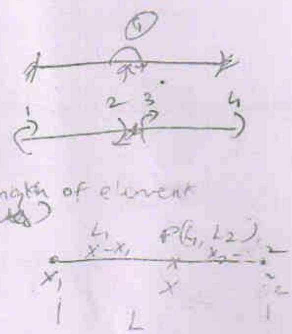
(x, y) no bending

$K = 80 \times 8 \quad \delta = 80 \times 1 \quad F = 80 \times 1$

~~Displacement functions~~ of

Co-ordinate system

1. Global co-ordinates →
2. Local co-ordinates →
3. Natural co-ordinates (Length of element)



Shape functions:

It is the one whose value at a particular node is unity at other nodes are zero. It relates field variable at any point within the element to the field variables of nodal point

One dimensional Polynomial shape functions

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \ldots$$

$$v = [G][\alpha]$$

$$G = [1, x, x^2, \ldots x^n]$$ – Geometric progression

$$[\alpha]^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ldots \alpha_{n+1}]$$ – constant

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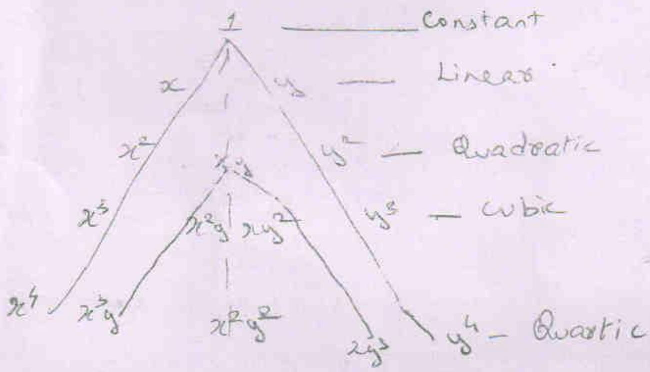
Two dimensional polynomial shape function

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \ldots$$

$$V(x, y) =$$

Convenient way ~~of~~ to remember two dimensional polynomial is in the form of Pascal Triangle



Pascal Triange

Three dimensional polynomial shape functions

$$u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 \cdot \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz + \ldots \alpha_m x^{n-1} z$$

Convenient way to remember three dimensional polynomial is tetrahedron

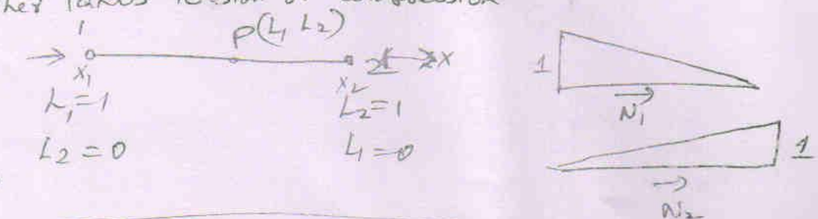
Derivation Shape functions N_1, N_2, N_3

shape function displacement $= u_1, u_2, u_3$

Can be expressed as $U = N_1 U_1 + N_2 U_2 + N_3 U_3 + \ldots$

Truss / Bar Element by natural co-ordinates

It either takes tension or compression



X - ho bar polynomial coordinates

$$U = \alpha_1 L_1 + \alpha_2 L_2$$

$$U_1 = \alpha_1 L_1$$

$$U_2 = \alpha_2 L_2$$

$$\text{or } U = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{bmatrix} \alpha_1 L_1 \\ \alpha_2 L_2 \end{bmatrix}$$

$$U = [L_1 \; L_2] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Write in matrix form

when $L_1 + L_2 = 1$

$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\therefore U = [L_1 \; L_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$= [L_1 \; L_2] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad — (1)$$

W.K.T

$$U = N_1 U_1 + N_2 U_2$$

$$= [N_1 \; N_2] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad — (2)$$

only two nodal values select polynomial with two constants

$$\therefore u = \alpha_1 + \alpha_2 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad -(1)$$

when $x = x_1$

$$u_1 = \alpha_1 + \alpha_2 x_1$$

when $x = x_2$

$$u_2 = \alpha_1 + \alpha_2 x_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 x_1 \\ \alpha_1 + \alpha_2 x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix}^{T} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Sub in (1) $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ since $x_2 - x_1 = \lambda$

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{\lambda} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u = \begin{bmatrix} \frac{x_2 - x}{\lambda} & \frac{x - x_1}{\lambda} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

W.K.T $u = N_1 u_1 + N_2 u_2 + \ldots$

$$= [N_1 \quad N_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [N_1 \quad N_2] \begin{Bmatrix} \text{displacement} \end{Bmatrix}$$

$$\therefore N_1 = \frac{x_2 - x}{\lambda} \qquad N_2 = \frac{x - x_1}{\lambda}$$

Compare ① & ②

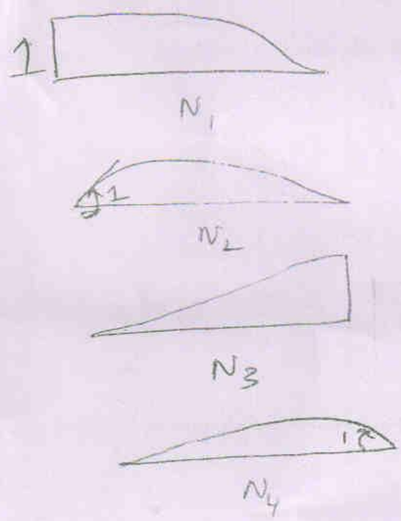
$N_1 = L_1 \qquad N_2 = L_2$

$N_1 = \frac{x_2 - x}{L} \qquad N_2 = \frac{x - x_1}{L}$

Shape functions for two beam noded element by polynomial functions

$x_1 = 0 \qquad x_2 = L$

$v_1, \theta_1 \qquad v_2, \theta_2$



w. k. T

$v_1 = w_1$

$\theta_1 = \frac{\partial w_1}{\partial x_1}$

δ = displacement

$$\delta = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

Since there are four nodes totally select Polynomial with four constants

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\theta = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

Since $x_1 = 0$

$$w_1 = \alpha_1$$

$$\theta_1 = \alpha_2$$

Since $x_2 = l$

$$w_2 = \alpha_1 + l\alpha_2 + l^2\alpha_3 + l^3\alpha_4$$

$$\theta_2 = \alpha_2 + 2l\alpha_3 + 3l^2\alpha_4$$

write in matrix form

$$\delta = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

$$= \frac{1}{3l^4 - 2l^4} \begin{bmatrix} l^4 & 0 & -3l^2 & 2l \\ 0 & l^4 & -2l^3 & l^2 \\ 0 & 0 & 3l^2 & -2l \\ 0 & 0 & -l^3 & l^2 \end{bmatrix}^T \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$ - skip

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Multiply by $\{1 \ x \ x^2 \ x^3\}$ on both sides

$$\{1 \ x \ x^2 \ x^3\} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = [1 \ x \ x^2 \ x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \left\{1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}, \quad x - \frac{2x^2}{l} + \frac{x^3}{l^2}, \quad \frac{3x^2}{l^2} - \frac{2x^3}{l^3}, \quad \frac{-x^2}{l} + \frac{x^3}{l^2}\right\} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

~~This is of the~~ We k.T $U = [N_1 \ N_2 \ N_3 \ N_4]\ \delta$

& this is of the above form

$\therefore$ ~~$N_1 N_2 N_3 N_4$~~

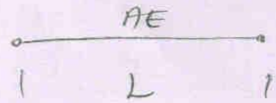
$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$N_4 = -x^2/l + x^3/l^2$$

Derivation of stiffness matrix for Bar element



Let unit +ve displacement at 1

$$K_{11} = \frac{EA}{L} \qquad K_{12} = -\frac{EA}{L}$$

Let unit +ve displacement at 2

$$K_{21} = -\frac{EA}{L}$$

$$K_{22} = \frac{EA}{L}$$

$$K = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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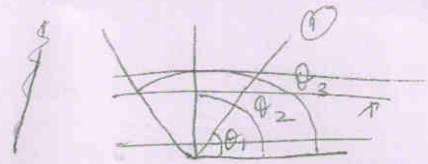
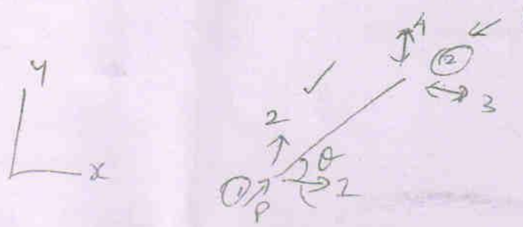
Element stiffness matrix for 2D Truss element





$\cos\theta = \frac{\delta}{\delta_1}$

$\therefore \delta \cos\theta = 1 \times \cos\theta$

Unit displacement of end 1 in x direction

$E = \frac{P/A}{\delta/L}$

$= \frac{PL}{A\delta} \quad \therefore P = \frac{EA\delta}{L}$

when unit displacement δ at P in that direction along inclined direction

$\therefore$ displacement along x axis $= \cos\theta = \frac{\delta_H}{\delta_i}$

$\therefore \cancel{P_H = \frac{EA}{L}\cos\theta}$ $\quad \therefore \delta_h = 1\cos\theta$

$\therefore$ Force P along Hzl direction $= P\cos\theta$

$$\boxed{\therefore P\cancel{\cos\theta} = \frac{EA\cos\theta}{L\cancel{\cos\theta}}}$$

$\therefore P_H = P\cos\theta = \frac{EA}{L}\cos^2\theta$

Element stiffness matrix

$K_{11} = P\cos\theta = \frac{EA}{L}\cos^2\theta$

$K_{21} = P\sin\theta = \frac{EA}{L}\cos\theta\sin\theta$

$$K_{31} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta$$

$$K_{41} = -P\sin\theta = -\frac{EA}{L}\cos\theta\sin\theta$$

Similarly by applying unit loads at 2, 3, 4 we get

$\boxed{\text{Short cut since the truss is like bar element}}$

find $K_{11}, K_{12}, K_{21}, K_{22}$ we have

$$K = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Here matrix 1 refers to $= \begin{matrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{matrix}$

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

(ii) unit displacement at 2

$$P = \frac{EA}{L}\sin\theta$$

$$K_{12} = P\cos\theta = \frac{EA}{L}\sin\theta\cos\theta$$

$$K_{22} = P\sin\theta = \frac{EA}{L}\sin^2\theta$$

$$K_{32} = -P\cos\theta = -\frac{EA}{L}\sin\theta\cos\theta$$

$$K_{42} = -P\sin\theta = -\frac{EA}{L}\sin^2\theta$$

(iii) unit displacement along 3

$$P = \frac{EA}{L}\cos\theta$$

$$K_{13} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta$$

Same as displacement stiffness at 1 but change sign

(iv) unit displacement along 4

$$P = \frac{EA}{L}\sin\theta$$

$$K_{14} = -P\cos\theta$$

Same as stiffness at 2 but change sign

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∴ Stiffness matrix is

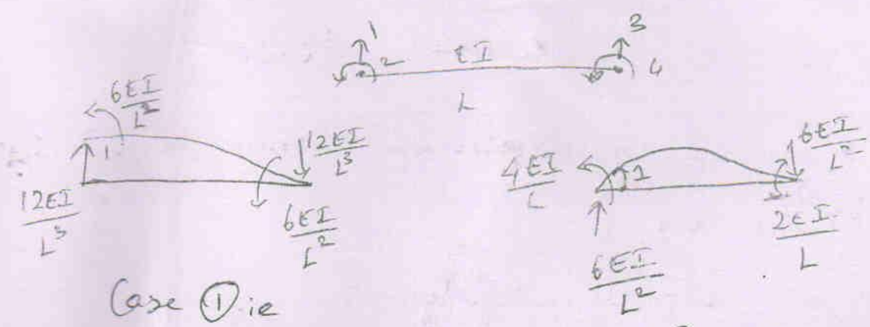
$$K = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & +\cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & +\cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Let $\cos\theta = l$ $\quad \sin\theta = m$

$$K = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & +l^2 & +lm \\ -lm & -m^2 & +lm & +m^2 \end{bmatrix}$$

Beam Element - 3D

Element stiffness matrix for 3D Beam Elements



Case ① ie
unit deflection at 1

Case 2
unit deflection at 2

rotations are on the same side - the

a. $K_{11} = \frac{12EI}{L^3}$ $K_{21} = \frac{6EI}{L^2}$ $K_{31} = -\frac{12EI}{L^3}$

$K_{41} = \frac{6EI}{L^2}$

b. $K_{21} = \frac{6EI}{L^2}$ $K_{22} = \frac{4EI}{L}$ $K_{32} = -\frac{6EI}{L^2}$ $K_{42} = \frac{2EI}{L}$

c. $K_{31} = -\frac{12EI}{L^3}$ $K_{32} = -\frac{6EI}{L^2}$ $K_{33} = \frac{12EI}{L^3}$ $K_{34} = \frac{-6EI}{L^2}$

~~c. Same as this~~ but opposite sign

d. $K_{41} = \frac{6EI}{L^2}$ $K_{42} = \frac{2EI}{L}$ $K_{34} = -\frac{6EI}{L^2}$ $K_{44} = \frac{4EI}{L}$

$$K = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

$$= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

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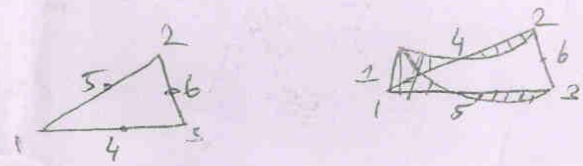
CST – Constant strain triangle (Three noded)

LST – Linear strain triangle (Six noded).



Problem 11.1 P428 Vaidi II

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II way:

$$\Delta = \text{Area of triangle} = \frac{1}{2}\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix}\begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}$$

where $b_i = y_j - y_k$ $\quad$ $b = y$ coordinates

$b_j = y_k - y_i$

$b_k = y_i - y_j$

$c_i = x_k - x_j$ $\quad$ $c = x$ co-ordinates

$c_j = x_i - x_k$

$c_k = x_j - x_i$

$$\Delta = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 100 & 0 \\ 1 & 0 & 80 \end{bmatrix} = 100 \times 80 = 8000$$

$b_i = 0 - 80 = -80$ $\quad$ $b_j = 80 - 0 = 80$ $\quad$ $b_k = 0$

$c_i = 0 - 100 = -100$ $\quad$ $c_j = 0$ $\quad$ $c_k = 100$

$$\varepsilon = \frac{1}{2 \times 4000}\begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 0 & -100 & 0 & 0 & 0 & 100 \\ -100 & -80 & 0 & 80 & 100 & 0 \end{bmatrix}\begin{bmatrix} 0.11 \\ 0.72 \\ 0.23 \\ 0.91 \\ 0.47 \\ 0.60 \end{bmatrix} = \begin{bmatrix} (-80 \times 0.11 + 80 \times 0.23)/8000 \\ (-100 \times 0.72 + 100 \times 0.60)/8000 \\ (-100 \times 0.11 - 0.8 \times 0.72 + 80 \times 0.91 + 100 \times 0.47)/8000 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0012 \\ 0.0015 \\ 0.0064 \end{bmatrix}$$

$$\sigma = D\,\varepsilon$$

$$\begin{matrix} \sigma_x \\ \sigma_y \\ \gamma_{xy} \end{matrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Same as above method I

University Problem

In a three noded CST element the displacement model for v is $v = ax + by + c$. The analysis show the nodal displacement as follows

$v_1 = 0 \qquad v_2 = 0.0172 \qquad v_3 = -0.010$

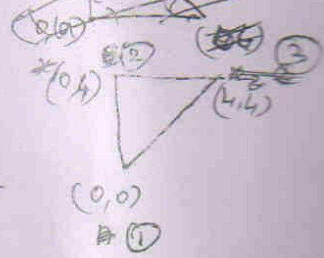
Find ε_x in the element with nodes $(0,0)\ (0,4)$ & $(4,4)$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\Delta u}{x_3 - x_2}$$

$$= \frac{u_3 - u_2}{x_3 - x_2}$$

$$= \frac{-0.010 - 0.0172}{4 - 0}$$

$$= -0.0068$$



Nodal Loads

In FEM, any loading on the continuum have to be converted into concentrated loads at the assigned co-ordinates.

Two methods of expressing distributed loads:

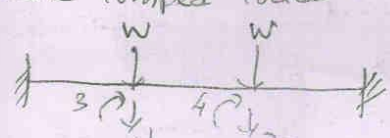
1. Direct method
2. Variational method

Direct method — Method of inspection / Equivalent load

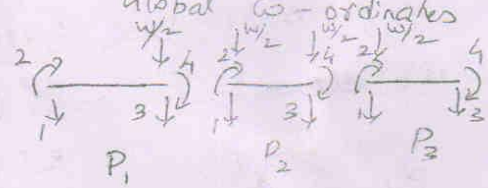
a. Method of inspection

Inspect the distributed loading & assign appropriate lumped loads

Ex. (1)



Global co-ordinates



$$P_1 = [0 \quad 0 \quad w/2 \quad 0]^T$$

$$P_2 = \left[\frac{w}{2} \quad 0 \quad w/2 \quad 0\right]^T$$

$$P_3 = [w/2 \quad 0 \quad 0 \quad 0]^T$$

$$F = [W \quad 0 \quad W \quad 0]^T$$

Nodal Loads

In FEM, any loading on the continuum have to be converted into concentrated loads at the assigned co-ordinates.

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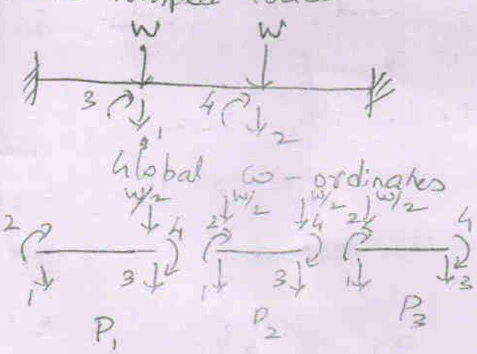
Ex. 1



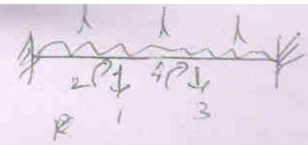
$$P_1 = [0 \quad 0 \quad w/2 \quad 0]^T$$

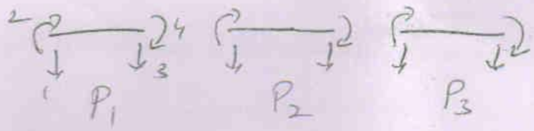
$$P_2 = \left[\frac{w}{2} \quad 0 \quad w/2 \quad 0\right]^T$$

$$P_3 = [\,w/2 \quad 0 \quad 0 \quad 0]^T$$

$$F = [w \quad 0 \quad w \quad 0]^T$$

(a) Ex 2



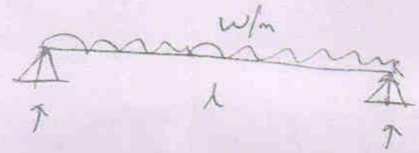


$$P_1 = \left[\frac{wl}{2} \quad 0 \quad \frac{wl}{2} \quad 0\right]^T = P_2 = P_3$$

$$F = \left[\frac{wl}{2}+\frac{wl}{2} \quad 0 \quad \frac{wl}{2}+\frac{wl}{2} \quad 0\right]$$

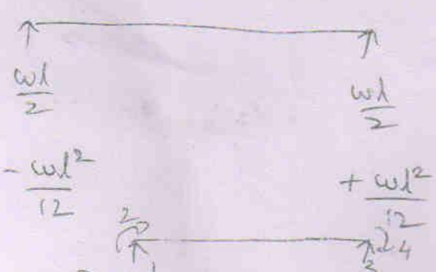
$$= \left[wl \quad 0 \quad wl \quad 0\right]$$

(b) Equivalent Loads



Reaction $\frac{wl}{2}$ $\frac{wl}{2}$

FEM $-\frac{wl^2}{12}$ $+\frac{wl^2}{12}$



$$\therefore P_1 = (-1)\left[\frac{wl}{2} \quad -\frac{wl^2}{12} \quad \frac{wl}{2} \quad \frac{wl^2}{12}\right]^T$$

$$= \left[-\frac{wl}{2} \quad \frac{wl^2}{12} \quad -\frac{wl}{2} \quad -\frac{wl^2}{12}\right]^T$$

Solve the matrix equation $F = [K][U]$

where $F^T = [100 \quad 120 \quad -10]$ $\quad K = \begin{bmatrix} 12 & 6 & 2 \\ 6 & 48 & 4 \\ 2 & 4 & 24 \end{bmatrix}$

Make sure that $U_1 = 0$ (or) Arrest the displacement at co ordinate 1

$$F = [K][U]$$

$$\begin{bmatrix} 100 \\ 120 \\ -10 \end{bmatrix} = \begin{bmatrix} 12 & 6 & 2 \\ 6 & 48 & 4 \\ 2 & 4 & 24 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

To make $U_1 = 0$, Replace F_{11} with 0
or to arrest displacement at 1 $U_1 = 0$

$$\begin{bmatrix} 0 \\ 120 \\ -10 \end{bmatrix} = \begin{bmatrix} 12 & 6 & 2 \\ 6 & 48 & 4 \\ 2 & 4 & 24 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$U_1 = 0$$

$$6U_1 + 48U_2 + 4U_3 = 120$$

$$48U_2 + 4U_3 = 120$$

$$2U_1 + 4U_2 + 24U_3 = -10$$

$$4U_2 + 24U_3 = -10$$

$$U = \begin{bmatrix} U_1 = 0 \\ U_2 = 2.43 \\ U_3 = -0.845 \end{bmatrix}$$

Above problem to have pre assigned value of $U_1 = 0.02$, Replace F_{11} with 0.02 of formula is

$$F = [K][U]$$

$$\begin{bmatrix} U_1 \\ F_2 - K_{12}U_1 \\ F_3 - K_{13}U_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\begin{bmatrix} 0.02 \\ 120 - 6\times0.02 \\ -10 - 2\times0.02 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 48 & 4 \\ 0 & 4 & 24 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$U_1 = 0.02$$

$$48U_2 + 4U_3 = 119.88$$

$$4U_2 + 24U_3 = -10.04$$

$$U_2 = 2.56$$

$$U_3 = 0.846$$

$$U = \begin{bmatrix} 0.02 \\ 2.56 \\ 0.846 \end{bmatrix}$$

Given that $U = 0.012 + 0.2x + 0.13x^2 + 0.05y + 0.005y^2 + 0.11xy$

a. Determine ε_x

b. Determine ε_x at three nodes of a triangle A (0,0), B (1,0), C (0,1)

Ans

a. $\varepsilon_x = \dfrac{\partial U}{\partial x} = 0.2 + 2\times0.13x + 0.11y$

$= 0.2 + 0.26x + 0.11y$

$$\begin{bmatrix} 0.02 \\ 120 - 6\times0.02 \\ -10 - 2\times0.02 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 48 & 4 \\ 0 & 4 & 24 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$v_1 = 0.02$$

$$48v_2 + 4v_3 = 119.88$$

$$4v_2 + 24v_3 = -10.04$$

$$v_2 = 2.56$$

$$v_3 = 0.846$$

$$v = \begin{bmatrix} 0.02 \\ 2.56 \\ 0.846 \end{bmatrix}$$

Given that $v = 0.012 + 0.2x + 0.13x^2 + 0.05y + 0.005y^2 + 0.11xy$

a. Determine ε_x

b. Determine ε_x at three nodes of a triangle A (0,0), B (1,0), C (0,1)

Ans

a. $\varepsilon_x = \dfrac{\partial v}{\partial x} = 0.2 + 2\times0.13x + 0.11y$

$= 0.2 + 0.26x + 0.11y$

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Ans b.

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0.2 + 0.26x + 0.11y$$

At node A $x = 0$ $y = 0$

$$\varepsilon_x = 0.2 + 0.26 \times 0 + 0.11 \times 0$$
$$= 0.2$$

At node B $x = 1$ $y = 0$

$$\varepsilon_x = 0.2 + 0.26 \times 1 + 0.11 \times 0$$
$$= 0.2 + 0.26 = 0.46$$

At node C, $x = 0$, $y = 1$

$$\varepsilon_x = 0.2 + 0.26 \times 0 + 0.11 \times 1$$
$$= 0.2 + 0.11 = 0.31$$

Plane stress + Plane strain elements

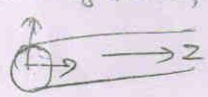
Plane stress:

The forces are predominant in plane (ie) x + y direction, + thickness negligible in z direction is a plane stress elements. eg. Thin plates subject to forces in their plane only.

Plane strain

The forces are predominant in lateral direction + very little forces in longitudinal direction.

eg. Pipes, long strip footings, retaining walls, gravity dams, tunnels etc



Pipe elongation in z direction is negligible but " " in circumferential direction is more + hence called plane strain element.

	Finite element method	classical methods
a.	Exact equations are formed but approximate solutions are obtained	Exact equations are formed & exact solutions are obtained.
b.	Solutions can be obtained for any complex situations	Solutions can be obtained for standard cases only
c.	No drastic assumptions for shape, boundary conditions, & loading	Drastic assumptions for shape, boundary conditions & loading
d.	It can handle structures with anisotropic property also (anisotropic - materials with varying properties)	It is difficult to handle anisotropic material.
e.	It can handle structures with different materials construction (or) composite materials	It is difficult to handle structures with more than one material.
f.	It can handle material non linearities & shape (geometry) non linearities.	Does not so & makes drastic assumptions.

	Finite element method	Finite difference method
a.	It ensures continuity along the grid lines also	It ensures continuity at the nodes only
b.	It gives values at any point by suitable shape or interpolation functions	It does not gives values except at nodes. i.e It gives values only at nodes.
c.	The curved & sloping boundaries can be handled exactly	It makes stair type approximation to curved of sloping boundaries.
d.	Good results are obtained with fewer nodes	Large number of nodes required to get good results.
e.	It can handle all complicated problems.	Most complicated problems are difficult to handle