

UNIT-I

FLEXIBILITY MATRIX METHOD

Equilibrium and compatibility - Determinate vs indeterminate structures - indeterminacy - primary structure - compatibility conditions - Analysis of indeterminate pin jointed plane frames, continuous Beams, rigid plane frames.

Introduction

A structure may be set to be in rest or in motion. The usual civil engineering structures say, a building, a dam or a bridge is all set at rest and hence may be said to be in statical equilibrium.

An in flight aeroplane or a moving train is in motion and hence is in dynamic equilibrium.

Static Equilibrium:-

The equation of static equilibrium are based on Sir Isaac Newton's law governing the motions of bodies which say

(a) The sum of all forces in any axis is zero

$$\text{i.e. } \sum F_x = 0; \quad \sum F_y = 0; \quad \sum F_z = 0$$

(b) The sum of all the moments about any axis is zero

$$\text{i.e. } \sum M_x = 0; \quad \sum M_y = 0; \quad \text{and } \sum M_z = 0 \quad \text{①}$$

Compatibility :-

In general, six independent condition of equilibrium exists. They are

- (a) Sum of the forces in three orthogonal directions is zero.
- (b) Sum of moments about the three orthogonal axes is zero.

Determinate vs indeterminate structures :-

* A structure is said to be statically determinate if the equations of static equilibrium (ie $\sum F=0$ and $\sum M=0$) are sufficient to determinate all the internal forces and moments in the members.

* A structure is said to be statically indeterminate if the equations of static equilibrium are unable to determine all the forces and moments in the members.

General Criteria for determining stability Determinacy and indeterminacy of a structural members:-

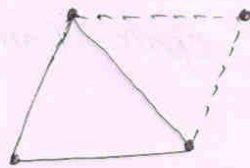
Very broadly, a planar structure may be a beam, a frame or a truss. A structure may be externally indeterminate or internally indeterminate.

A structure said to be externally indeterminate if the available equations of equilibrium are not sufficient to determinate the external reaction.

(i) plane truss

$$m = 3 + 2(j-3)$$

$$= 2j - 3$$

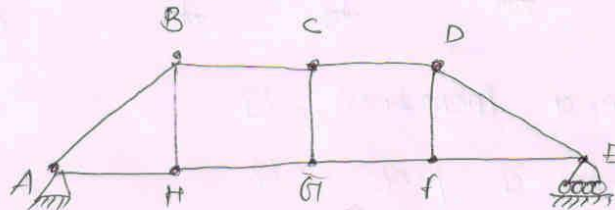


* If a simple plane truss has $m < (2j-3)$, the truss is unstable.

* If $m > (2j-3)$ the truss should be examined for stability and internal determinacy.

* If $m = (2j-3)$ the truss is a necessary but not sufficient condition for stability.

Stable and determinate structures:-



(2)

Number of joints = 8

Number of members = 13

Number of Reactions = 9

External Indeterminate (i) = $2j - 3 - m$

$$= (2 \times 8 - 3) - 13$$

$$= 13 - 13$$

$$= 0$$

Space frame:-

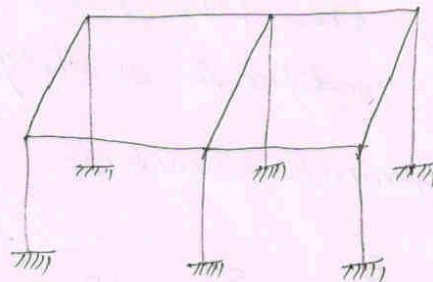
At each joint of a space frame six equations can be written:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

$$\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0$$

$$6j = 6m + r$$

$$i = (6m + r) - 6j$$



(Indeterminate frame structure)

Number of members = 13

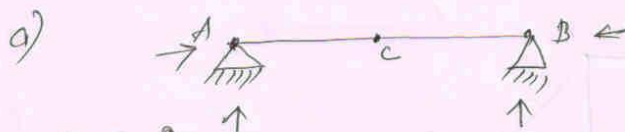
Number of joints = 12

Each support has six unknowns reaction

$$r = 6 \times 6 = 36$$

$$\begin{aligned} \therefore i &= (6m+r) - 6j \\ &= (6 \times 13 + 36) - 6 \times 12 \\ &= (78 + 36) - 72 \\ &= 42 \end{aligned}$$

Check the external and internal conditions and determinacy of the plane trusses shown in Fig.



$$m = 2$$

$$j = 3$$

$$r = 4$$

$$m+r = 2j$$

$$m+r = 6$$

$$2j = 6$$

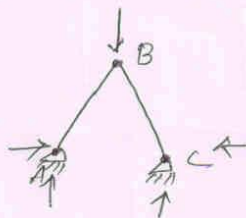
$$2 \times 3 = 6$$

$$6 = 6$$

$$\underline{i = 0}$$

stable & determinate.

b)



$$m = 2$$

$$j = 3$$

$$r = 4$$

$$m+r = 2j$$

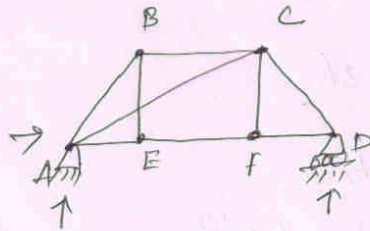
$$2+4 = 2 \times 3$$

$$i = 0$$

stable & determinate

③

c)



$$m = 9$$

$$j = 6$$

$$r = 3$$

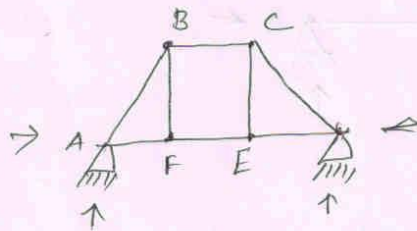
$$m + r = 2j$$

$$9 + 3 = 2 \times 6$$

$$12 = 12$$

Stable & determinate.

d)



$$m = 8$$

$$j = 6$$

$$r = 4$$

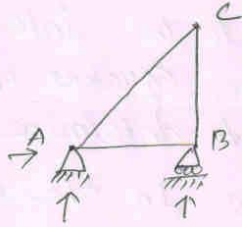
$$m + r = 2j$$

$$8 + 4 = 2 \times 6$$

$$12 = 12$$

Stable & determinate

e)



$$m = 3$$

$$j = 3$$

$$r = 3$$

$$m + r = 6$$

$$3 + 3 = 6$$

$$6 = 6$$

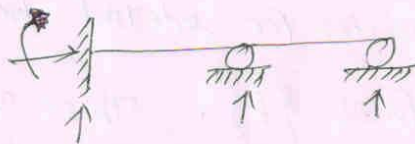
stable and determinate.

Indeterminacy:-

The indeterminacy of a structure may be external
(or) internal (or) both.

External indeterminacy:-

A structure is said to be externally indeterminate (or) externally redundant if the equations of statics (ie $\sum V = 0$, $\sum H = 0$ and $\sum M = 0$) are not sufficient to determine all the unknown reactions acting on the structure.



$$R = 5, \quad I = 3$$

$$= 5 - 3$$

$$= 2$$

(4)

Internal indeterminacy

A structure is said to be internally indeterminate or internally redundant, if the equations of static equilibrium are not sufficient to determine all the internal forces and moments in the members.

Matrix Method (Flexibility)

The flexibility method also known as the force method or compatibility method.

Primary Structure

If the structure given is not initially determinate, it has to be made determinate, by introducing sufficient releases such as hinges and cuts. The structure so reduced is called the primary structure.

8 steps procedure for Flexibility Method:-

- 1] Decide on the primary structures; indicate redundants $\{F\}^o$.
- 2] Select $\{P\}$ Co-ordinates for internal forces.
- 3] Compile external forces $\{f\}^*$. This would include the effects of off node forces.

Then $\{F\} = \left\{ \frac{\{f\}^*}{\{F\}^0} \right\}$ and $u \begin{bmatrix} \{u\}^* \\ \{u\}^0 \end{bmatrix}, \{u\}^0 = \{0\}$

Generate $[b]$ matrix such that
 $\{P\} = [b] \{F\}$; $[b] = \begin{bmatrix} [b]^* \\ [b]^0 \end{bmatrix}$

4) Synthesize elements flexibility matrix $[\alpha]$

5] Compute $[\alpha]$ using

$$[a] = [b]^T [\alpha] [b]; \quad [a] = \begin{bmatrix} [a]_{11} & [a]_{12} \\ [a]_{21} & [a]_{22} \end{bmatrix}$$

6] Isolate $\{F\}^0$, the redundant forces, from the

Condition

$$\{u\}^0 = \{0\}$$

$$\{F\}^0 = -[a]_{22}^{-1} [a]_{21} \{F\}^*$$

7] Isolate $[a]^*$

$$[a]^* = [a]_{11} - [a]_{12} [a]_{22}^{-1} [a]_{21}$$

8] Get $\{P\}$ from

$$\{P\} = [b] \{F\}$$

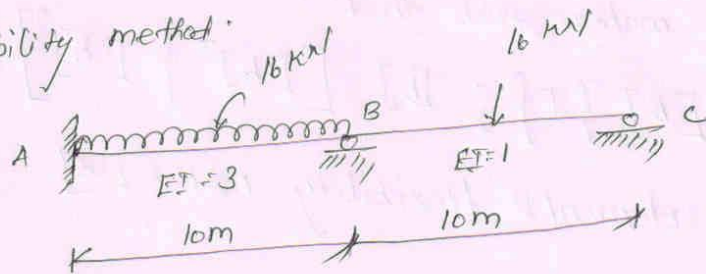
$$\text{and } \{P\}^F = \{P\} - \{P\}^e$$

$\{P\}^F$ are the final member forces.

Example 1

Analyse the Continuous beam in Fig. by

Flexibility method.

Solution

No. of unknowns = 5

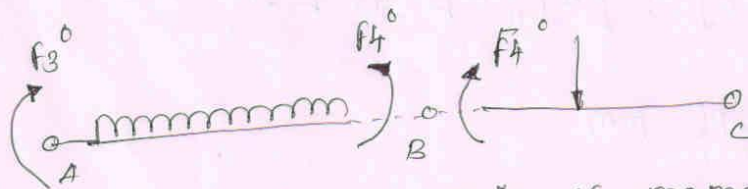
Available eqn = 3

degree of redundancy = $5 - 3$
= 2

∴ The structure has 2 degree of redundancy.

Step: 1

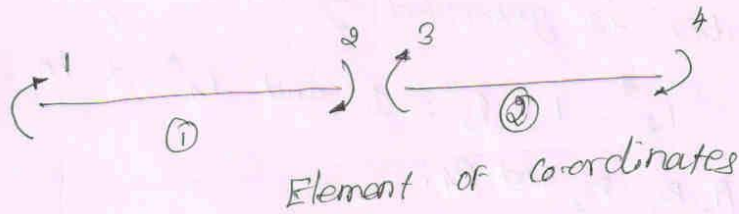
The primary structure is chosen as below by introducing hinges at A and B. The moments at A and B are the redundants.



please note that F_4^0 is a pair of moments making a bending moment.

Step:2

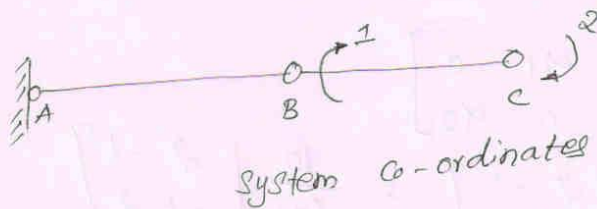
The $\{P\}$ system is as under



Step:3

The $\{F\}$ system is in 2 parts, $\{F\}^*$ and $\{F\}^0$

F^* is primarily made up of the effects of the lateral loads on AB and BC. These are the opposites of fixed end moments.



Span	FEM	Name of moment	Equivalent joints loads
AB	$-\frac{wl^2}{12} = -133.33$	M_{AB}	133.33
	$+\frac{wl^2}{12} = +133.33$	M_{BA}	-133.3
BC	$-\frac{wl}{8} = -20.0$	M_{BC}	20
	$+\frac{wl}{8} = +20.0$	M_{CB}	-20

(6)

Hence $f_1^* = -113.30 \text{ kN}\cdot\text{m}$

$f_2^* = -20.0 \text{ kN}\cdot\text{m}$

The $[b]$ matrix is generated by introducing

$f_1^* = 1, f_2^* = 1, f_3^0 = 1$ and $f_4^0 = 1$ in steps

and finding P_1, P_2, P_3 and P_4 .

Hence $[b] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$[b]^* \quad [b]^0$

The assembled element flexibility matrix $[\alpha]$ is

$[\alpha] = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$

$[\alpha_1] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{10}{6 \times 3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$[\alpha_2] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{10}{6 \times 1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$[\alpha] = \begin{bmatrix} \frac{10}{18} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} & 0 & 0 \\ 0 & 0 & \frac{10}{6} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{bmatrix}$

$$= \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}$$

$$a = [b]^T [\alpha] [b]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b^*{}^T \\ b^0{}^T \end{bmatrix} [\alpha] \begin{bmatrix} b^* & b^0 \end{bmatrix}$$

$$a_{22} = a_{00} = b^0{}^T [\alpha] b^0$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$a_{00} = \frac{10}{18} \begin{bmatrix} 2 \\ 18 \end{bmatrix}$$

$$[a_{00}]^{-1} = 0.12 \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[a_{0*}] = [b_0]^T [\alpha] [b_*]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 0 & 0 \\ 6 & -3 \end{bmatrix}$$

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$$\{F\}^0 = -[a_{00}]^{-1} [a_0^*] \{f\}^*$$

$$= 0.12 \begin{bmatrix} 8 & -1 \\ -1 & 2 \end{bmatrix} \frac{10}{18} \begin{bmatrix} 0 & 0 \\ 6 & -3 \end{bmatrix} \begin{Bmatrix} -113.3 \\ -20.0 \end{Bmatrix}$$

$$\{F\}^0 = \begin{Bmatrix} -41.3 \\ 82.7 \end{Bmatrix}$$

$$\{F\}^T = [-113.3 \quad -20 \quad -41.3 \quad 82.7]$$

The element forces $\{P\}$ are given by

$$\{P\} = [b] \{F\}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -113.3 \\ -20.0 \\ -41.3 \\ 82.7 \end{Bmatrix} = \begin{Bmatrix} -41.3 \\ -82.7 \\ -30.6 \\ -20 \end{Bmatrix}$$

The final forces $\{P\}^f$ given by

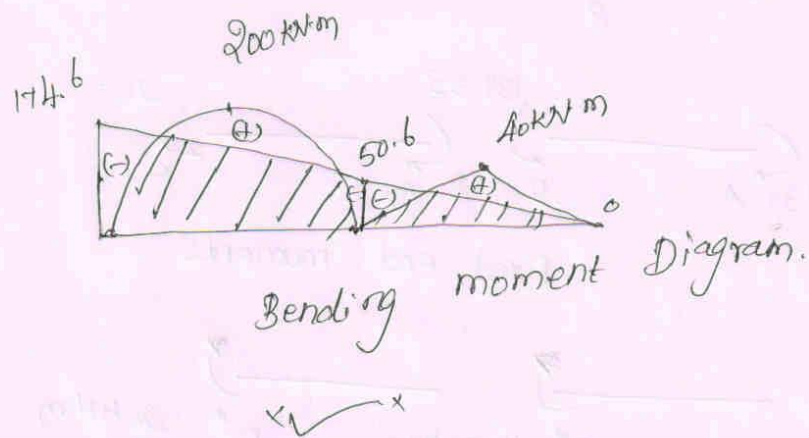
$$\{P\}^f = \{P\} - \{P\}^e$$

$$\{P\}^f = \begin{Bmatrix} -41.3 \\ -82.7 \\ -30.6 \\ -20 \end{Bmatrix} - \begin{Bmatrix} 133.3 \\ -133.3 \\ 20 \\ -20 \end{Bmatrix} = \begin{Bmatrix} -174.6 \\ 50.6 \\ -50.6 \\ 0 \end{Bmatrix}$$

Free BMD:-

$$M_{AB} = \frac{wl^2}{8} = \frac{16 \times 100}{8} = 200 \text{ kN}\cdot\text{m}$$

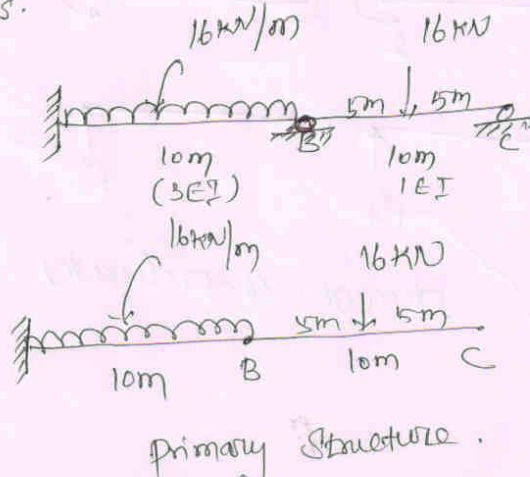
$$BC = \frac{wl}{4} = \frac{16 \times 10}{4} = 40 \text{ kN}\cdot\text{m}$$



Example: 2

Analyse the beam in the example 1 by treating the reactions at B and C as redundants. Obtain the support moments.

Solution



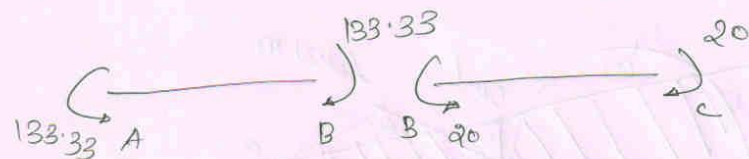
fixed end moments

$$M_{AB} = -\frac{wL^2}{12} = -133.33 \text{ kN}\cdot\text{m}$$

$$M_{BA} = +\frac{wL^2}{12} = 133.33 \text{ kN}\cdot\text{m}$$

$$M_{BC} = -\frac{wL^2}{8} = -20 \text{ kN}\cdot\text{m}$$

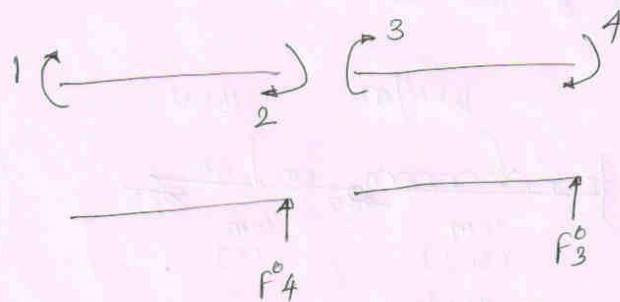
$$M_{CB} = +\frac{wL^2}{8} = 20 \text{ kN}\cdot\text{m}$$



fixed end moments.

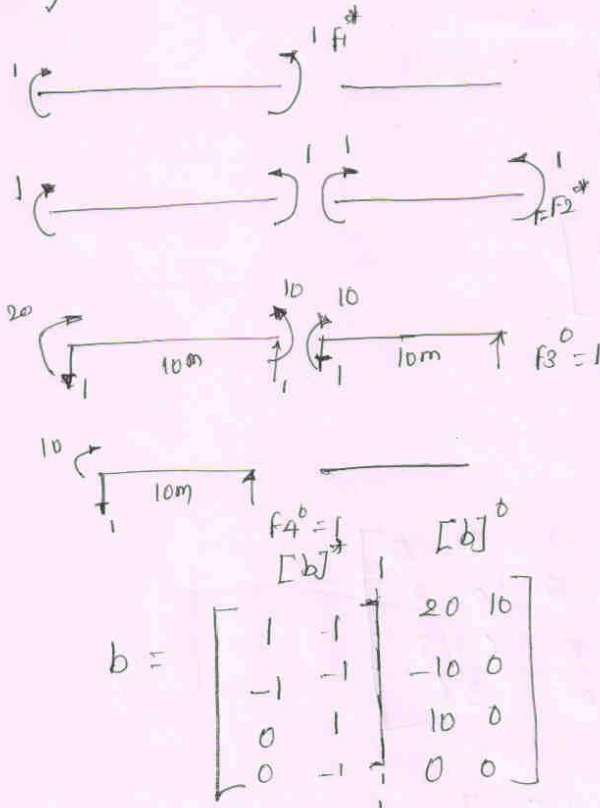


System co-ordinates.



Element co-ordinates

Applying f_1^* , f_2^* , f_3^* and f_4^*



$$b = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 20 & 10 \\ -10 & 0 \\ 10 & 0 \\ 0 & 0 \end{bmatrix}$$

Element flexibility matrix

$$[k] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore [k_1] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{10}{6 \times 3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[\alpha_2] = \frac{\Delta_2}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{10}{6} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

$$[\alpha] = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$

$$[\alpha] = \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix}$$

$$[a] = [b]^T [\alpha] [b]$$

$$a_{00} = [b^0]^T [\alpha] [b^0]$$

$$= \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \times \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 20 & 10 \\ -10 & 0 \\ 10 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 50 & 26 \\ -40 & -10 \\ 60 & 0 \\ -30 & 0 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 2000 & 500 \\ 500 & 400 \end{bmatrix}$$

$$[a_{00}]^{-1} = \frac{18}{10} \times \frac{1}{(200 \times 200) - (500 \times 500)} \begin{bmatrix} 200 & -500 \\ -500 & 2000 \end{bmatrix}$$

$$= \frac{18 \times 100}{10 \times 150000} \begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix} = \frac{18}{15000} \begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix}$$

$$[a_0^*] = [b^0]^T [a] [b^*]$$

$$= \begin{bmatrix} 20 & -10 & 10 & 0 \\ +10 & 0 & 0 & 0 \end{bmatrix} \times \frac{10}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 20 & -10 & 10 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & 3 \\ 0 & 9 \\ 0 & -9 \end{bmatrix}$$

$$= \frac{10}{18} \begin{bmatrix} 90 & 180 \\ 30 & 30 \end{bmatrix}$$

Redundant forces are given by

$$\{f^0\} = -[a_{00}]^{-1} [a_0^*] \{f^*\}$$

$$= \frac{18}{15000} \begin{bmatrix} 2 & -5 \\ -5 & 20 \end{bmatrix} \times \frac{10}{18} \begin{bmatrix} 90 & 180 \\ 30 & 30 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \end{Bmatrix}$$

$$= \frac{-1}{15000} \begin{bmatrix} 30 & 210 \\ 150 & -300 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \end{Bmatrix}$$

$$= \frac{-1}{15000} \begin{Bmatrix} 7599.9 \\ 10999.5 \end{Bmatrix}$$

$$\{F^0\} = \begin{Bmatrix} -5.067 \\ -7.333 \end{Bmatrix}$$

$$\{P\} = [b] \{F\}$$

$$\{P\} = \begin{bmatrix} 1 & 1 & 20 & 10 \\ -1 & -1 & -10 & 0 \\ 0 & 1 & 10 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 113.33 \\ 20 \\ -5.067 \\ -7.333 \end{Bmatrix}$$

$$= \begin{Bmatrix} -41.34 \\ -82.66 \\ -30.67 \\ -20 \end{Bmatrix}$$

Final force

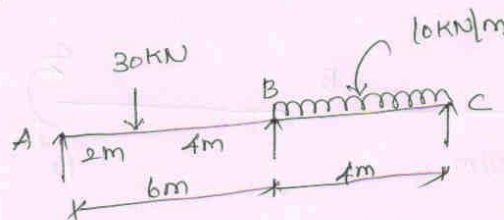
$$\{P\}^F = \begin{Bmatrix} -133.33 \\ 133.33 \\ -20 \\ +20 \end{Bmatrix} + \begin{Bmatrix} -41.34 \\ -82.66 \\ -30.67 \\ -20.00 \end{Bmatrix}$$

$$= \begin{Bmatrix} 174.67 \\ 50.67 \\ -50.67 \\ 0 \end{Bmatrix}$$

← X

Example: 3

Analyse the Continuous Beam shown in Fig. By the flexibility method and draw the bending moment diagram.



Solution :-

No. of Reaction :- 4

No. of Equilibrium :- 3

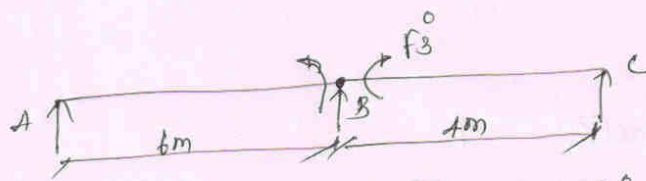
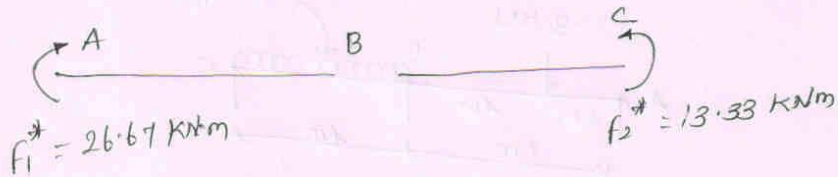
Degree of Redundancy : $4 - 3 = 1$

Fixed end moment

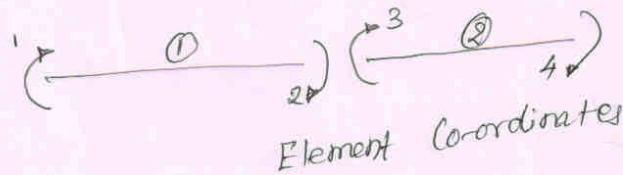
Span	Fixed end moments	Name of moment	Equivalent joint force (kN/m)
AB	$-\frac{wab^2}{l^2} = -26.67$	M_{AB}	26.67
	$\frac{wa^2b}{l^2} = 13.33$	M_{BA}	-13.33
BC	$-\frac{wl^2}{12} = -13.33$	M_{BC}	13.33
	$\frac{wl^2}{12} = 13.33$	M_{CB}	-13.33

(11)

$$\{F\}^* = \begin{Bmatrix} 26.67 \\ -13.33 \end{Bmatrix} \text{ and } \{f\} = \begin{Bmatrix} \{F\}^* \\ \{f\}^0 \end{Bmatrix}$$

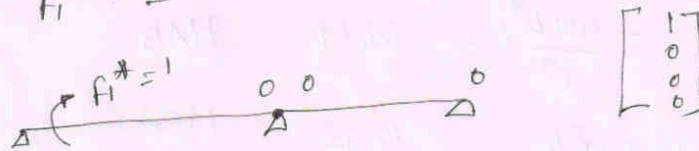


Primary Structure f_3^0

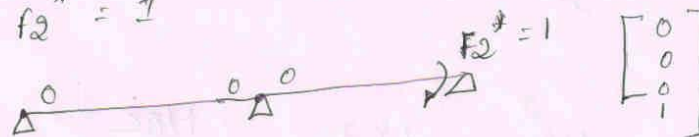


formation of $[b]$ Matrix:-

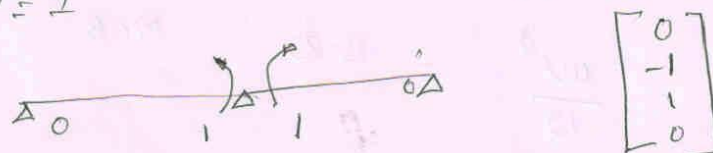
Applying $F_1^* = 1$



Applying $f_2^* = 1$



Applying $f_3^* = 1$



Hence $[b]$ Matrix: -

$$[b] = \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & -1 \\ 0 & 0 & | & +1 \\ 0 & 1 & | & 1 \end{bmatrix}$$

Element Flexibility Matrix $[\alpha]$:-

$$[\alpha_1] = \frac{L_1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{6}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[\alpha_2] = \frac{4}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1.33 & -0.67 \\ -0.67 & 1.33 \end{bmatrix}$$

$$\alpha = \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix}$$

TO Find Redundant Forces

$$\{F\}^0 = -[a^{00}]^{-1} [a^{0*}] \{F\}^*$$

$$[a^{00}] = [b^0]^T [\alpha] [b^0]$$

$$[a^{0*}] = [b^0]^T [\alpha] [b^*]$$

(12)

$$[a_{00}] = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} \times \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & -2 & 1.33 & -0.67 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI} [3.33]$$

$$[a_{00}] = \frac{3.33}{EI} ; [a_{00}]^{-1} = \frac{EI}{3.33}$$

$$[a_0^*] = [b^0]^T [a] [b^*]$$

$$= \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} \times \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1.33 & -0.67 \\ 0 & 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & -2 & 1.33 & -0.67 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} [1 - 0.67]$$

$$\therefore \{f\}^0 = - \left[\frac{EI}{3.33} \right] \times \frac{1}{EI} [1 - 0.67] \begin{Bmatrix} 26.67 \\ -19.33 \end{Bmatrix}$$

$$\{f\}^0 = -10.69 \text{ kN-m}$$

Element force

$$\{P\} = [b] \{F\}$$

$$\{P\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & +1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 26.67 \\ -13.33 \\ -10.69 \end{Bmatrix}$$

$$= \begin{Bmatrix} 26.67 \\ 10.69 \\ -10.69 \\ -13.33 \end{Bmatrix} \text{ kN.m}$$

$$\{P\}^f = \{P\} - \{P\}^e$$

$$= \begin{Bmatrix} 26.67 \\ 10.69 \\ -10.69 \\ -13.33 \end{Bmatrix} - \begin{Bmatrix} 26.67 \\ -13.33 \\ +13.33 \\ -13.33 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 24.02 \\ -24.02 \\ 0 \end{Bmatrix}$$

Support moments:

$$M_{AB} = \frac{w_{ab} l}{2} = \frac{30 \times 2 \times 4}{6} = 40 \text{ kN.m}$$

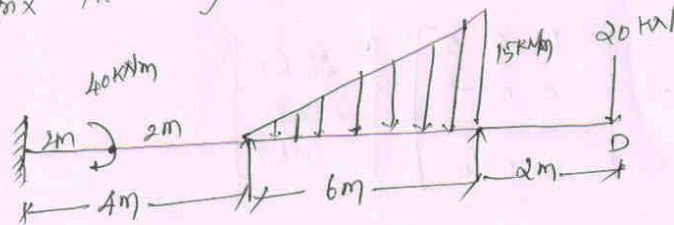
$$M_{BC} = \frac{w_{bc} l^2}{8} = \frac{10 \times 4^2}{8} = 20 \text{ kN.m}$$



BMD

Example : 4

Analyse the continuous beam loaded as shown in fig. by the matrix flexibility method and sketch BMD

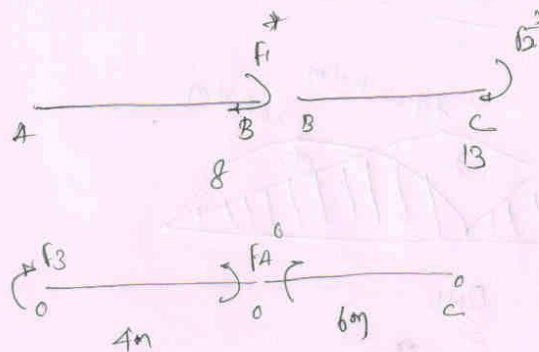


Solution

The degree of Indeterminacy is 2.

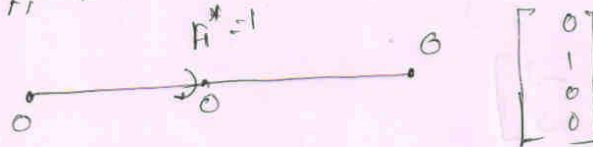
Fixed end moments

Span	Fixed end moments	Name	Equivalent joint force (kNm)
AB	$\frac{M}{4} = 10 \text{ kN}\cdot\text{m}$	M_{AB}	$-10 \text{ kN}\cdot\text{m}$
	$\frac{M}{4} = 10 \text{ kN}\cdot\text{m}$	M_{BA}	$-10 \text{ kN}\cdot\text{m}$
BC	$\frac{-wL^2}{20} = -18 \text{ kN}\cdot\text{m}$	M_{BC}	$+18 \text{ kN}\cdot\text{m}$
	$\frac{wL^2}{20} = +27 \text{ kN}\cdot\text{m}$	M_{CB}	$+27 \text{ kN}\cdot\text{m}$
	$-20 \times 2 = -40 \text{ kN}\cdot\text{m}$	M_{CD}	$40 \text{ kN}\cdot\text{m}$

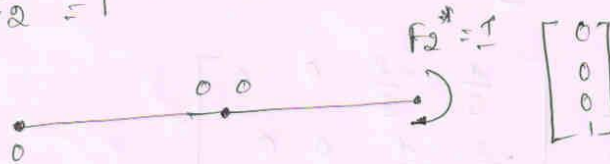


formation of $[b]$ matrix:-

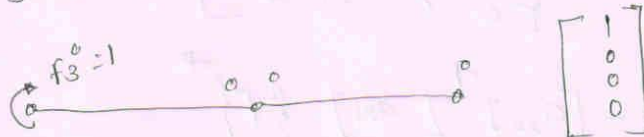
Applying $F_1^* = 1$



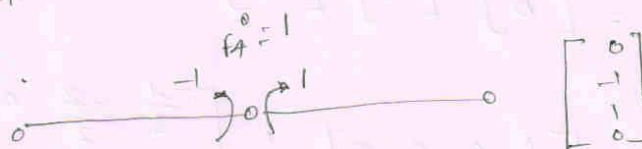
Applying $F_2^* = 1$



Applying $F_3^0 = 1$



Applying $F_4^0 = 1$



Hence b matrix is given by

$$[b] = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

b^* b^0

Element flexibility matrix is

$$[k] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[k_1] = \frac{4}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

$$[a_{22}] = \frac{6}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[\alpha] = \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\{F\}^0 = -[a_{22}]^{-1} [a_{21}] \{F\}^*$$

$$[a_{22}] = [b^0]^T [\alpha] [b^0]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[a_{22}] = \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{10}{3} \end{bmatrix}$$

$$[a_{21}] = [b^0]^T [\alpha] [b^*]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[a_{21}] = \frac{1}{EI} \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -\frac{2}{3} & 0 \\ -\frac{4}{3} & -1 \end{bmatrix}$$

$$\{f_0\} = -[a_2]^{-1} [a_{21}] \{f\}^*$$

$$= -\frac{EI}{4} \begin{bmatrix} \frac{10}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \times \frac{1}{EI} \begin{bmatrix} -\frac{2}{3} & 0 \\ -\frac{4}{3} & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 7.0 \end{bmatrix}$$

Element forces

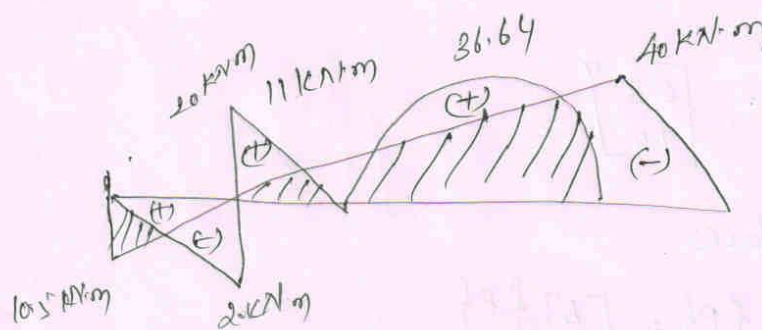
$$\{P\} = [b] \{F\}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 13 \\ 0.5 \\ 7.0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 7 \\ 13 \end{bmatrix}$$

Final forces $[P]^f = [P] - [P]^e$

$$= \begin{bmatrix} 0.5 \\ 10 \\ 7.0 \\ 13.0 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \\ 18 \\ -27 \end{bmatrix}$$

$$= \begin{bmatrix} 10.5 \\ 11 \\ -11 \\ 40 \end{bmatrix}$$

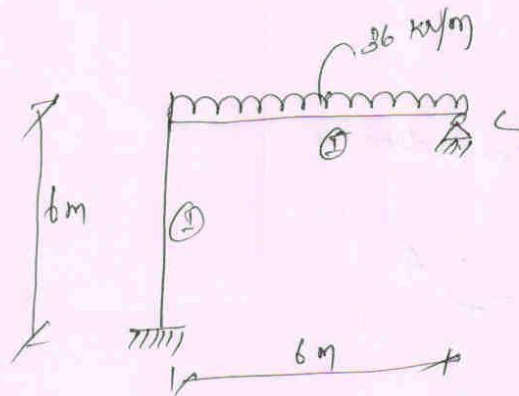


BND



Example: 5

Analyse the given frame using matrix flexibility method.



Q.1

Degree of indeterminacy = 2

fixed end moments

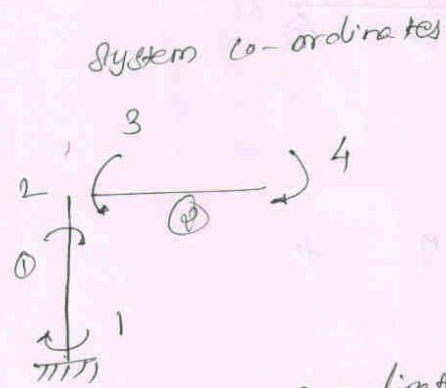
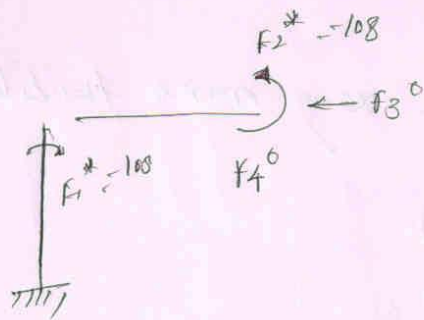
$$M_{AB} = M_{BA} = 0$$

$$M_{BC} = \frac{-wL^2}{12} = -108 \text{ kN}\cdot\text{m}$$

$$M_{CB} = 108 \text{ kN}\cdot\text{m}$$

$$\{f\} = \begin{bmatrix} F_1^* \\ F_2^* \\ F_3^0 \\ F_4^0 \end{bmatrix}$$

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Element Co-ordinates.

Element flexibility matrix.

$$[k_1] = [k_2] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{6}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[K] = \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Formation of $[b]$ matrix

$$f_1^* = 1$$



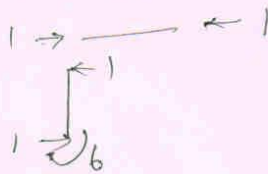
$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f_2^* = 2$$



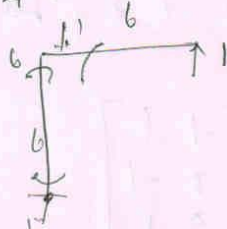
$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$f_3^0 = 1$$



$$\begin{bmatrix} 6 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

$$f_4^0 = 1$$



$$\begin{bmatrix} 6 \\ -6 \\ 6 \\ 0 \end{bmatrix}$$

Redundant forces

$$\{F\}^0 = -[a_{22}]^{-1} [a_{21}] \{f\}^T$$

$$[a_{22}] = [b_0]^T [a] [b]$$

$$= \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 0 & -6 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 12 & -6 & 0 & 0 \\ 18 & -18 & 12 & -6 \end{bmatrix} \begin{bmatrix} 6 & 6 \\ 0 & -6 \\ 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$[a_{22}] = \frac{1}{EI} \begin{bmatrix} 72 & 108 \\ 108 & 288 \end{bmatrix}$$

$$[a_{21}] = [b_0]^T [a] [b^*]$$

$$= \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 12 & -6 & 0 & 0 \\ 18 & -18 & 12 & -6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$[q_{21}] = \frac{1}{EI} \begin{bmatrix} -18 & 18 \\ -36 & 54 \end{bmatrix}$$

$$\{f\}^0 = -EI \begin{bmatrix} 72 & 108 \\ 108 & 288 \end{bmatrix}^{-1} \frac{1}{EI} \begin{bmatrix} -18 & 18 \\ -36 & 54 \end{bmatrix} \begin{bmatrix} 108 \\ 108 \end{bmatrix}$$

$$= \frac{1}{9072} \begin{bmatrix} 288 & -108 \\ -108 & 72 \end{bmatrix} \begin{bmatrix} 0 \\ 1944 \end{bmatrix}$$

$$\{f_0\} = \begin{Bmatrix} 23.14 \\ -15.43 \end{Bmatrix}$$

Element force :-

$$\{P\} = [b] \{F\}$$

$$= [b^*] [b^0] \begin{Bmatrix} f^* \\ f_0 \end{Bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 6 & 6 \\ 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 6 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 108 \\ 108 \\ 23.14 \\ -15.43 \end{bmatrix}$$

$$= \begin{Bmatrix} 46.26 \\ 92.58 \\ 15.24 \\ -108 \end{Bmatrix}$$

Final forces

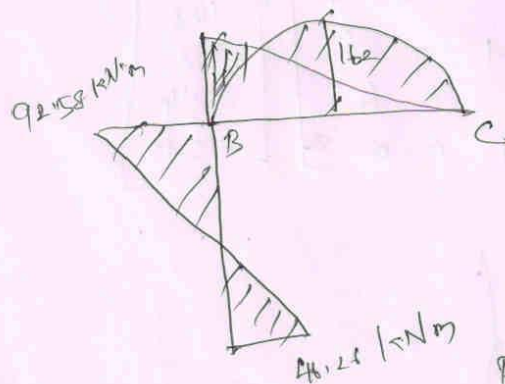
$$\{P^f\} = \{P\} + \text{fixed end moments}$$

$$= \begin{Bmatrix} 46.26 \\ 92.58 \\ 15.42 \\ -108 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -108 \\ 108 \end{Bmatrix}$$

$$= \begin{Bmatrix} 46.26 \\ 92.58 \\ -92.58 \\ 0 \end{Bmatrix}$$

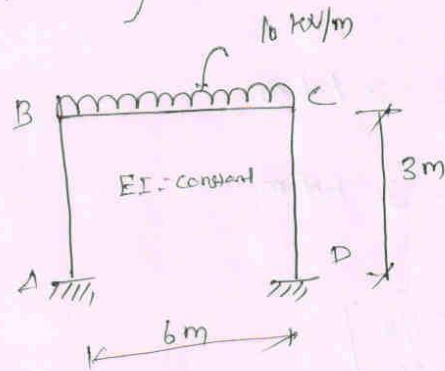
Support moments

$$\text{Span } BC = \frac{Wl^2}{8} = \frac{36 \times 6^2}{8} = 162 \text{ kN/m}$$



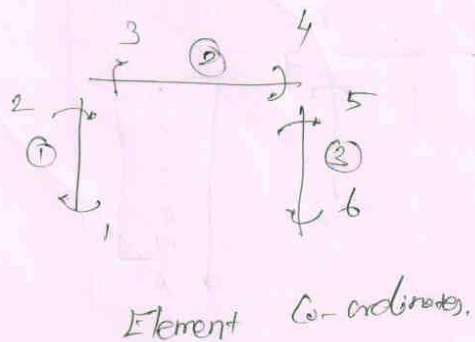
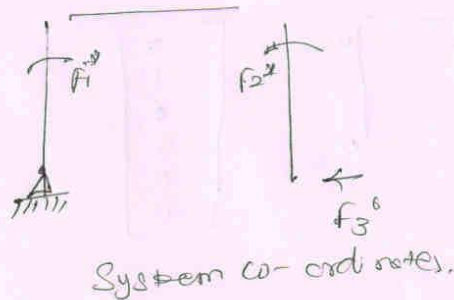
Example: 6

Analyse the portal frame loaded as shown in fig.
by matrix flexibility method and sketch BMD.



Solution

Degree of indeterminacy 1



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fixed end moments :-

$$M_{AB} = M_{BA} = M_{CD} = M_{DC} = 0$$

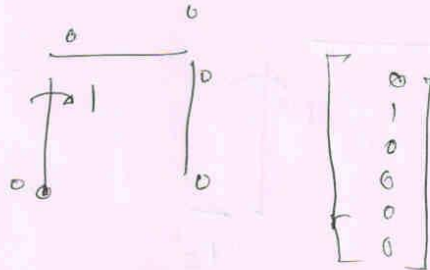
$$M_{BC} = \frac{-10 \times 6^2}{12} = -30 \text{ kN.m}$$

$$M_{CB} = \frac{10 \times 6^2}{12} = 30 \text{ kN.m}$$

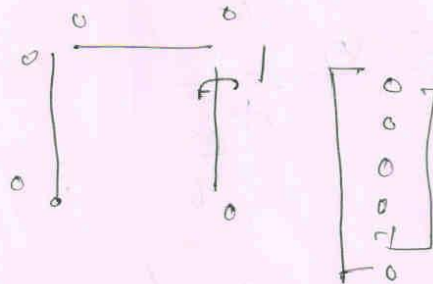
$$\begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix} = \begin{bmatrix} 30 \\ -30 \end{bmatrix}$$

formation of $[b]$ matrix :-

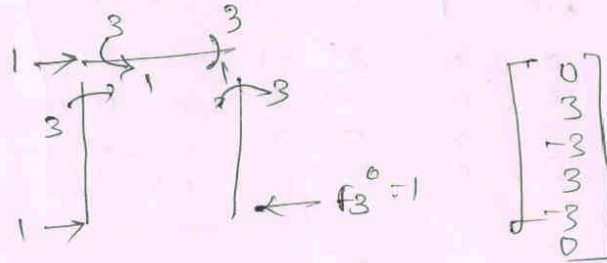
Applying $F_1^* = 1$



Applying $F_2^* = 1$



Applying $F_3^0 = 1$



$$[b] = \begin{array}{c} b^* \quad b^0 \\ \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

Element Flexibility Matrix $[Q]$

$$[Q] = \frac{1}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Q_1 = Q_3 = \frac{3}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$Q_2 = \frac{6}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

$$\{f_0\} = -[q_{22}]^{-1} [q_{21}] \{F^*\}$$

$$[q_{22}] = [b^0]^T [\alpha] [b^0]$$

$$= [0 \ 3 \ -3 \ 3 \ -3 \ 0] \frac{1}{EI} \begin{bmatrix} 1 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -3 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{EI} [-1.5 \ 3 \ -9 \ 9 \ -3 \ 1.5] \begin{bmatrix} 0 \\ 3 \\ -3 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

$$[q_{22}] = \left[\frac{72}{EI} \right]$$

$$[q_{22}]^{-1} = \frac{EI}{72}$$

$$[q_{21}] = [b^0]^T [\alpha] [b^*]$$

Final forces:-

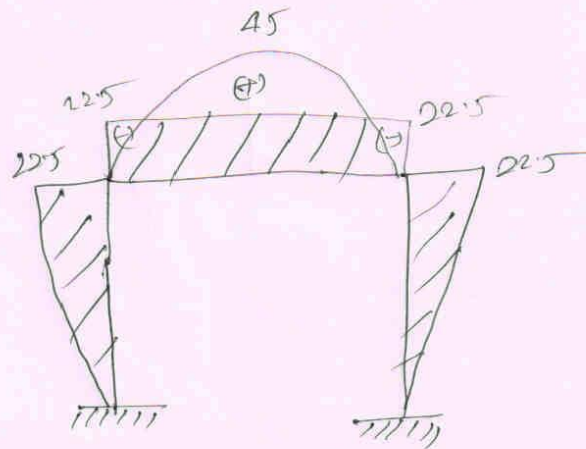
$$\{P\} = [b] \{f\}$$

$$= \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ -2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 22.5 \\ 7.5 \\ -7.5 \\ -22.5 \\ 0 \end{bmatrix}$$

$$\{P^f\} = [P] - \{P^e\}$$

$$= \begin{bmatrix} 0 \\ 22.5 \\ -22.5 \\ 22.5 \\ 22.5 \\ 0 \end{bmatrix}$$



BMD.

← x

(21)