Unit - I

Solutions of Equations and Eigen Value Problems.

Iterative Method :

1. Write the gn eqn $f(x) = 0$ into the form $x = \varphi(x)$
2. Assume that $x = x_0$ be the root of the given eqn
3. The first approximation to the root is gn by $x_1 = \varphi(x_0)$

|||ly
$$x_2 = \varphi(x_1)$$
$$x_3 = \varphi(x_2)$$
$$\vdots$$
$$x_n = \varphi(x_{n-1})$$

n th iteration + the value of

$\Rightarrow x_n$ is the the root of the gn eqn.

x_n is the root of

1. Find the root of the equation $\cos x = 3x - 1$, using iteration Method.

Soln

$$f(x) = \cos x - 3x + 1$$
$$f(0) = \cos 0 - 3(0) + 1 = 2 \rightarrow +ve$$
$$f(1) = \cos 1 - 3(1) + 1 = 0 - 3(1) + 1 \rightarrow -ve$$

$\therefore$ The root lies between 0 and 1

The eqn can be written as

$$\cos x - 3x + 1 = 0$$
$$-3x = -\cos x - 1$$
$$3x = \cos x + 1$$
$$x = \frac{1}{3}[1 + \cos x]$$

Let $\varphi(x) = \frac{1}{3}[1 + \cos x]$

$$\varphi'(x) = -\frac{1}{3}\sin x$$
$$|\varphi'(x)| = \frac{1}{3}\sin x$$

$$|\varphi'(0)| = 0 < 1$$
$$|\varphi'(1)| = \frac{1}{3}\sin 1 = 0.2804 < 1.$$

Let $x_0 = 0$

$$x_1 = \varphi(x_0) = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + \cos 0)$$
$$x_1 = 0.6667$$
$$x_2 = \varphi(x_1) = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos 0.6667)$$
$$x_2 = 0.5953$$
$$x_3 = \varphi(x_2) = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos 0.5953)$$
$$x_3 = 0.6093$$
$$x_4 = \varphi(x_3) = \frac{1}{3}(1 + \cos x_3) = \frac{1}{3}(1 + \cos 0.6093)$$
$$x_4 = 0.6067$$

$x_5 = \varphi(x_4) = \frac{1}{3}(1+\cos x_4) = \frac{1}{3}(1+\cos 0.6067)$

$x_5 = 0.6072$

$x_6 = \varphi(x_5) = \frac{1}{3}(1+\cos x_5) = \frac{1}{3}(1+\cos 0.6072)$

$x_6 = 0.6071$

$x_7 = \varphi(x_6) = \frac{1}{3}(1+\cos x_6) = \frac{1}{3}(1+\cos 0.6071)$

$x_7 = 0.6071$

$\therefore$ The required root is 0.6071.

(2) Solve the equation $x^2 - 2x - 3 = 0$ by iteration method. for the +ve root

Soln

$$x^2 - 2x - 3 = 0$$

$$f(x) = x^2 - 2x - 3$$

$f(0) = 0 - 2(0) - 3 = -3 \to -ve$

$f(1) = -4 \to -ve$

$f(2) = -3 \to -ve$

$f(3) = 0 \to +ve$

$\therefore$ The root lies between 2 and 3

$$x^2 = 2x + 3$$

$$x = \sqrt{2x+3}$$

$$\varphi(x) = \sqrt{2x+3} = (2x+3)^{1/2}$$

$$\varphi'(x) = \frac{1}{2}(2x+3)^{-1/2} \cdot 2$$

$$|\varphi'(x)| = |(2x+3)^{-1/2}|$$

$$|\varphi'(2)| < 1 \quad \& \quad |\varphi'(3)| < 1$$

Take $x_0 = 2.5$

$x_1 = \varphi(x_0) = \sqrt{2x_0+3} = \sqrt{2(2.5)+3} = 2.8284$

$x_2 = \varphi(x_1) = \sqrt{2x_1+3} = \sqrt{2(2.8284)+3} = 2.9422$

$x_3 = \varphi(x_2) = \sqrt{2x_2+3} = \sqrt{2(2.9422)+3} = 2.9807$

$x_4 = \varphi(x_3) = \sqrt{2x_3+3} = \sqrt{2(2.9807)+3} = 2.9936$

$x_5 = \varphi(x_4) = \sqrt{2x_4+3} = \sqrt{2(2.9936)+3} = 2.9979$

$x_6 = \varphi(x_5) = \sqrt{2x_5+3} = \sqrt{2(2.979)+3} = 2.9993$

$x_7 = \varphi(x_6) = \sqrt{2x_6+3} = \sqrt{2(2.9993)+3} = 2.9998$

$x_8 = \varphi(x_7) = \sqrt{2x_7+3} = \sqrt{2(2.9998)+3} = 2.9999$

$x_9 = \varphi(x_8) = \sqrt{2x_8+3} = \sqrt{2(2.9999)+3} = 2.9999$

$\therefore$ The required root is 2.9999

(3) Solve by iteration Method $2x - \log_{10} x = 7$

Soln

$$2x - \log_{10} x - 7 = 0$$

$$f(x) = 2x - \log_{10} x - 7$$

$f(1) = -5 \rightarrow -ve$

$f(2) = -3.3010 \rightarrow -ve$

$f(3) = -1.4771 \rightarrow -ve$

$f(4) = 0.3979 \rightarrow +ve$

$\therefore$ The root lies between 3 and 4

$$2x = 7 + \log_{10} x$$

$$x = \frac{1}{2}\left[7 + \log_{10} x\right]$$

$$\therefore \phi(x) = \frac{1}{2}\left[7 + \log_{10} x\right]$$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x} \log_{10} e\right]$$

$$|\phi'(x)| = \left|\frac{1}{2}\left[\frac{1}{x} \log_{10} e\right]\right| < 1 \text{ in } (3, 4)$$

Take $x_0 = 3.6$

$$x_1 = \phi(x_0) = \frac{1}{2}\left[\log_{10} x_0 + 7\right]$$

$$= \frac{1}{2}\left[\log_{10} 3.6 + 7\right]$$

$$= 3.7782$$

$$x_2 = \varphi(x_1) = \frac{1}{2}[\log_{10} x_1 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7782 + 7]$$
$$x_2 = 3.7886$$
$$x_3 = \varphi(x_2) = \frac{1}{2}[\log_{10} x_2 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7886 + 7]$$
$$x_3 = 3.7892$$
$$x_4 = \varphi(x_3) = \frac{1}{2}[\log_{10} x_3 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7892 + 7]$$
$$x_4 = 3.7893$$
$$x_5 = \varphi(x_4) = \frac{1}{2}[\log_{10} x_4 + 7]$$
$$= \frac{1}{2}[\log_{10} 3.7893 + 7]$$
$$x_5 = 3.7893$$

$\therefore$ The required root is 3.7893

H.W 4) find the negative root of the eqn $x^3 - 2x + 5 = 0$

Gauss Jordan Method

① $2x - y + 6z = 22$
$x + 7y - 3z = -22$
$5x - 2y + 3z = 18$

Soln

$$[A, B] = \left[\begin{array}{ccc|c} 2 & -1 & 6 & 22 \\ 1 & 7 & -3 & -22 \\ 5 & -2 & 3 & 18 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 3 & 11 \\ 1 & 7 & -3 & -22 \\ 1 & \frac{-2}{5} & \frac{3}{5} & \frac{18}{5} \end{array}\right] \quad R_1 \to \frac{R_1}{2}, \quad R_3 \to \frac{R_3}{5}$$

$$= \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 3 & 11 \\ 0 & \frac{15}{2} & -6 & -33 \\ 0 & \frac{1}{10} & \frac{-12}{5} & \frac{-37}{5} \end{array}\right] \quad R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & -6 & -22 \\ 0 & 1 & \frac{-4}{5} & \frac{-22}{5} \\ 0 & 1 & -24 & -74 \end{array}\right] \quad R_1 \to -2R_1, \quad R_2 \to \frac{2}{15}R_2, \quad R_3 \to 10R_3$$

$$= \left[\begin{array}{cccc} -2 & 0 & \frac{-26}{5} & \frac{-88}{5} \\ 0 & 1 & \frac{-4}{5} & \frac{-22}{5} \\ 0 & 0 & \frac{-116}{5} & \frac{-348}{5} \end{array}\right] \quad R_1 \to R_1 - R_2, \quad R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} \frac{5}{13} & 0 & 1 & \frac{44}{13} \\ 0 & -\frac{5}{4} & 1 & \frac{11}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 \to R_1 \times \frac{-5}{26} \\ R_2 \to -\frac{5}{4} R_2 \\ R_3 \to -\frac{5}{116} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{5}{13} & 0 & 0 & \frac{5}{13} \\ 0 & -\frac{5}{4} & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_2 \to R_2 \times \frac{-4}{5} \\ R_1 \to R_1 \times \frac{13}{5} \end{array}$$

$x = 1$, $y = -2$, $z = 3$.

(2) Solve
$$x + 3y + 3z = 16$$
$$x + 4y + 3z = 18$$
$$x + 3y + 4z = 19$$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1/3 & 1 & 1 & 16/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 \to \frac{R_1}{3}$$

$$= \left[\begin{array}{ccc|c} 1/3 & 0 & 1 & 10/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 \to R_1 - R_2$$

$$= \left[\begin{array}{ccc|c} 1/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 \to R_1 - R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] R_1 \to 3R_1$$

$x = 1, \quad y = 2, \quad z = 3$

(3) Solve

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$x + y + 5z = 7$$

Soln

$$[A, B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1/10 & 1/10 & 6/5 \\ 1 & 5 & 1/2 & 13/2 \\ 1 & 1 & 5 & 7 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{10} \\ R_2 \to \frac{R_2}{2} \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & \frac{1}{10} & \frac{1}{10} & \frac{6}{5} \\ 0 & \frac{49}{10} & \frac{2}{5} & \frac{53}{10} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{29}{5} \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc} 10 & 1 & 1 & 12 \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 1 & \frac{49}{9} & \frac{58}{9} \end{array}\right] \begin{array}{l} R_1 \to 10 R_1 \\ R_2 \to \frac{10}{49} R_2 \\ R_3 \to \frac{10}{9} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 10 & 0 & \frac{45}{49} & \frac{535}{49} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & \frac{2365}{441} & \frac{2365}{441} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{98}{9} & 0 & 0 & \frac{107}{9} \\ 0 & \frac{49}{4} & 1 & \frac{53}{4} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 \to \frac{49}{45} R_1 \\ R_2 \to \frac{49}{4} R_2 \\ R_3 \to \frac{441}{2365} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} \frac{98}{9} & 0 & 0 & \frac{98}{9} \\ 0 & \frac{49}{4} & 0 & \frac{49}{4} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 \times \frac{9}{98} \\ R_2 \to R_2 \times \frac{4}{49} \end{array}$$

$x = 1 \quad y = 1 \quad z = 1$

Inverse of a Matrix

Gauss Jordan Method

(1) find the inverse of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ using Gauss Jordan Method.

soln

$$A = \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{array}\right)$$

$$(A/I) = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{1} \\ R_3 \to \frac{R_3}{-2} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & -1/2 \end{array}\right] R_2 \to$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1/2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 1 & -1 & -1 & 0 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{2} \\ R_3 \to \frac{R_3}{1} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 2 & -1/2 & -1/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1/6 & 0 & 1 & 1/4 & -1/12 & 0 \\ 0 & -1/3 & 1 & 1/6 & -1/6 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/6 \\ R_2 \to R_2/-3 \\ R_3 \to R_3/2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1/6 & 0 & 0 & 1/2 & 1/6 & -1/4 \\ 0 & -1/3 & 0 & 5/12 & 1/12 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 \times 6 \\ R_2 \to -3R_2 \end{array}$$

$$= [I / A]$$

$\therefore$ Inverse of A is $\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$

(2) find the inverse of the Matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss Jordan Method.

Soln

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ -1 & 4 & 10 \end{bmatrix}$$

$$[A/I] = \left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ -1 & 4 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -\frac{6}{15} & +\frac{1}{3} & 0 & -\frac{1}{15} & 0 \\ 1 & -4 & -10 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{3} \\ R_2 \to \frac{R_2}{-15} \\ R_3 \to \frac{R_3}{-1} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{0}{3} & -\frac{1}{3} & -\frac{1}{15} & 0 \\ 0 & -\frac{11}{3} & -\frac{31}{3} & -\frac{1}{3} & 0 & -1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -3 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 10 & 5 & 1 & 0 \\ 0 & 1 & \frac{31}{11} & \frac{1}{11} & 0 & \frac{3}{11} \end{array}\right] \begin{array}{l} R_1 \to R_1(-3) \\ R_2 \to -15R_2 \\ R_3 \to \frac{-3}{11} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -3 & 0 & -1 & -6 & -1 & 0 \\ 0 & 1 & 10 & 5 & 1 & 0 \\ 0 & 0 & -\frac{79}{11} & -\frac{54}{11} & -1 & \frac{3}{11} \end{array}\right] \begin{array}{l} R_3 \to R_3 - R_2 \\ R_1 \to R_1 - R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} +3 & 0 & 1 & 6 & 1 & 0 \\ 0 & \frac{1}{10} & 1 & \frac{1}{2} & \frac{1}{10} & 0 \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{-1} \\ R_2 \to \frac{R_2}{10} \\ R_3 \to \frac{11}{31} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & \frac{185}{31} & 1 & -\frac{3}{31} \\ 0 & \frac{1}{10} & 0 & \frac{29}{62} & \frac{1}{10} & -\frac{3}{31} \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{185}{93} & \frac{1}{3} & \frac{-1}{31} \\ 0 & 1 & 0 & \frac{290}{62} & 1 & \frac{-30}{31} \\ 0 & 0 & 1 & \frac{1}{31} & 0 & \frac{3}{31} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{3} \\ R_2 \to R_2 \times 10 \end{array}$$

$= [I \mid A]$

$\therefore$ inverse of A is $\begin{bmatrix} \frac{85}{93} & \frac{1}{3} & \frac{-1}{31} \\ \frac{290}{62} & 1 & \frac{-30}{31} \\ \frac{1}{31} & 0 & \frac{3}{31} \end{bmatrix}$

3) Using Gauss Jordan Method find the inverse of $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$$

$$(A/I) = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & \frac{4}{3} & \frac{5}{3} & 0 & \frac{1}{3} & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{1} \\ R_2 \to \frac{R_2}{3} \\ \cancel{B} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{4}{3} & \frac{8}{3} & -1 & \frac{1}{3} & 0 \\ 0 & -6 & -7 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{\cancel{2}} & \frac{-3}{4} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & \frac{}{0} & 0 & -\frac{1}{6} \end{array}\right] \begin{array}{l} R_2 \to \frac{3}{4} R_2 \\ R_3 \to \frac{R_3}{-6} \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & \cancel{\text{[illegible]}} & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{\cancel{5}}{\cancel{6}} & \frac{3}{4} & -\frac{1}{4} & \frac{-1}{6} \end{array}\right] R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{-1} \\ R_2 \to \frac{R_2}{2} \\ R_3 \to -\frac{6}{5} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -\frac{1}{10} & -\frac{3}{10} & -\frac{1}{5} \\ 0 & \frac{1}{2} & 0 & \frac{21}{40} & -\frac{7}{40} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{21}{20} & -\frac{7}{20} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{array}\right]$$

$$= [\, I / A \,]$$

Inverse of A is $\begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & -\frac{7}{20} & -\frac{2}{5} \\ -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$

Gauss Jacobi Method

① Solve the following eqns by Gauss Jacobi Method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$x = \frac{17 - y + 2z}{20}$	$y = \frac{-18 + z - 3x}{20}$	$z = \frac{25 - 2x + 3y}{20}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 0.85$	$y_1 = -0.9$	$z_1 = 1.25$
$x_2 = 1.02$	$y_2 = -0.965$	$z_2 = 1.03$
$x_3 = 1.0013$	$y_3 = -1.0015$	$z_3 = 1.0033$
$x_4 = 1.0004$	$y_4 = -1.0001$	$z_4 = 0.9996$
$x_5 = 0.9999$	$y_5 = -1.0001$	$z_5 = 0.9999$
$x_6 = 1$	$y_6 = -1$	$z_6 = 1$
$x_7 = 1$	$y_7 = -1$	$z_7 = 1$

$\therefore \; x = 1, \; y = -1, \; z = 1$

② Solve
$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

$x = \frac{32-4y+z}{28}$	$y = \frac{35-4x-2x}{17}$	$z = \frac{24-x-3y}{10}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 1.1429$	$y_1 = 2.0588$	$z_1 = 2.4$
$x_2 = 0.9345$	$y_2 = 1.3597$	$z_2 = 1.6681$
$x_3 = 1.0082$	$y_3 = 1.5564$	$z_3 = 1.898$
$x_4 = 0.9883$	$y_4 = 1.4935$	$z_4 = 1.8323$
$x_5 = 0.9949$	$y_5 = 1.514$	$z_5 = 1.8531$
$x_6 = 0.9931$	$y_6 = 1.5058$	$z_6 = 1.847$
$x_7 = 0.9937$	$y_7 = 1.5074$	$z_7 = 1.8490$
$x_8 = 0.9936$	$y_8 = 1.5069$	$z_8 = 1.8484$
$x_9 = 0.9936$	$y_9 = 1.5070$	$z_9 = 1.8486$
$x_{10} = 0.9936$	$y_{10} = 1.5070$	$z_{10} = 1.8485$
$x_{11} = 0.9936$	$y_{11} = 1.5070$	$z_{11} = 1.8485$

$\therefore$ The soln is

$x = 0.9936 \quad y = 1.5070 \quad z = 1.8485$

(3) Solve
$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

$x = \frac{85 - 6y + z}{27}$	$y = \frac{72 - 6x - 2z}{15}$	$z = \frac{110 - x - y}{54}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.148$	$y_1 = 4.8$	$z_1 = 2.037$
$x_2 = 2.157$	$y_2 = 3.269$	$z_2 = 1.890$
$x_3 = 2.492$	$y_3 = 3.685$	$z_3 = 1.937$
$x_4 = 2.401$	$y_4 = 3.545$	$z_4 = 1.923$
$x_5 = 2.432$	$y_5 = 3.583$	$z_5 = 1.927$
$x_6 = 2.423$	$y_6 = 3.570$	$z_6 = 1.926$
$x_7 = 2.426$	$y_7 = 3.574$	$z_7 = 1.926$
$x_8 = 2.425$	$y_8 = 3.573$	$z_8 = 1.926$

$x = 2.425 \quad y = 3.573 \quad z = 1.926.$

Gauss Seidal Iteration Method

(1) Solve

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Soln:

$x = \frac{17-y+2z}{20}$	$y = \frac{-18-3x+z}{20}$	$z = \frac{25-2x+3y}{20}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 0.82$	$y_1 = -1.0275$	$z_1 = 1.0109$
$x_2 = 1.0025$	$y_2 = -0.9998$	$z_2 = 0.9998$
$x_3 = 1.0000$	$y_3 = -1.000$	$z_3 = 1.0000$
$x_4 = 1.0000$	$y_4 = -1.0000$	$z_4 = 1.0000$

$x = 1$ $y = -1$ $z = 1$

(2) Solve

$$4x + 2y + z = 14$$
$$x + 5y - z = 10$$
$$x + y + 8z = 20$$

$x = \frac{85-6y+z}{27}$	$y = \frac{72-6x-2z}{15}$	$z = \frac{110-x-y}{54}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.148$	$y_1 = 3.541$	$z_1 = 1.913$
$x_2 = 2.432$	$y_2 = 3.572$	$z_2 = 1.926$
$x_3 = 2.426$	$y_3 = 3.573$	$z_3 = 1.926$
$x_4 = 2.426$	$y_4 = 3.573$	$z_4 = 1.926$

$\therefore$ $x = 2.426$

$y = 3.573$

$z = 1.926$.

$x = \frac{14-2y-z}{4}$	$y = \frac{10-x+z}{5}$	$z = \frac{20-x-y}{8}$
$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$x_1 = 3.5$	$y_1 = 1.3$	$z_1 = 1.9$
$x_2 = 2.375$	$y_2 = 1.905$	$z_2 = 1.965$
$x_3 = 2.056$	$y_3 = 1.982$	$z_3 = 1.995$
$x_4 = 2.010$	$y_4 = 1.997$	$z_4 = 1.999$
$x_5 = 2.002$	$y_5 = 1.999$	$z_5 = 2$
$x_6 = 2.001$	$y_6 = 2$	$z_6 = 2$
$x_7 = 2$	$y_7 = 2$	$z_7 = 2$
$x_8 = 2$	$y_8 = 2$	$z_8 = 2$

$\therefore \; x = 2, \; y = 2, \; z = 2$

(3) Solve

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Eigen Values of a Matrix by power Method

(1) Find the numerically largest eigen value of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $(1\ 0\ 0)^T$ (upto three decimal places.

Soln

(1) Given $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 1 \\ 2 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = 25X_2$$

$$AX_2 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 25.2 \\ 1.12 \\ 0.08 \end{pmatrix} = 25.2\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix}$$

$$= 25.2X_3$$

$$AX_3 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0444 \\ 0.0667 \end{pmatrix} = \begin{pmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{pmatrix} = 25.1778\begin{pmatrix} 1 \\ 0.0450 \\ 0.0688 \end{pmatrix}$$

$$= 25.1778X_4$$

$$AX_4 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0450 \\ 0.06888 \end{pmatrix} = \begin{pmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{pmatrix}$$

$$= 25.1826\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1826 X_5$$

$$AX_5 = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = \begin{pmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{pmatrix}$$

$$= 25.1821\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix} = 25.1821 X_6$$

Dominant eigen value $\lambda = 25.1821$

corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.0451 \\ 0.0685 \end{pmatrix}$

(2) Determine by Power method the largest eigen value and the corresponding eigen vector of the Matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

Soln

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 0.3333 \\ 1 \\ -0.333 \end{bmatrix} = 3X_2$$

$$AX_2 = \begin{bmatrix} 3.6666 \\ 1.6667 \\ 0.3337 \end{bmatrix} = 3.6666\begin{bmatrix} 1 \\ 0.4546 \\ 0.0910 \end{bmatrix} = 3.6666\, X_3$$

$$AX_3 = \begin{bmatrix} 2.2728 \\ 4.2732 \\ 1.7284 \end{bmatrix} = 4.2732\begin{bmatrix} 0.5319 \\ 1 \\ 0.4045 \end{bmatrix} = 4.2732\, X_4$$

$$AX_4 = \begin{bmatrix} 3.1274 \\ 5.2137 \\ 7.5131 \end{bmatrix} = 7.5131\begin{bmatrix} 0.4163 \\ 0.6939 \\ 1 \end{bmatrix} = 7.5731\, X_5$$

$$AX_5 = \begin{bmatrix} 1.498 \\ 6.6367 \\ 12.3593 \end{bmatrix} = 12.3593\begin{bmatrix} 0.1212 \\ 0.5370 \\ 1 \end{bmatrix} = 12.3596\, X_6$$

$$AX_6 = \begin{bmatrix} 0.7322 \\ 5.4376 \\ 12.0268 \end{bmatrix} = 12.0268\begin{pmatrix} 0.0609 \\ 0.4521 \\ 1 \end{pmatrix} = 12.0268\, X_7$$

$$AX_7 = \begin{pmatrix} 0.4172 \\ 5.0869 \\ 11.7473 \end{pmatrix} = 11.7475 \begin{pmatrix} 0.0355 \\ 0.4330 \\ 1 \end{pmatrix} = 11.7475 X_8$$

$$AX_8 = \begin{pmatrix} 0.3345 \\ 4.9725 \\ 11.6965 \end{pmatrix} = 11.6965 \begin{pmatrix} 0.0286 \\ 0.4251 \\ 1 \end{pmatrix} = 11.6965 X_9$$

$$AX_9 = \begin{pmatrix} 0.3039 \\ 4.936 \\ 11.6718 \end{pmatrix} = 11.6718 \begin{pmatrix} 0.0260 \\ 0.4229 \\ 1 \end{pmatrix} = 11.6718 \begin{pmatrix} 0.0260 \\ 0.4229 \\ 1 \end{pmatrix}$$

$$AX_{10} = \begin{pmatrix} 0.2967 \\ 4.9238 \\ 11.6656 \end{pmatrix} = 11.6656 \begin{pmatrix} 0.0253 \\ 0.4221 \\ 1 \end{pmatrix} = 11.6656 X_{11}$$

$$AX_{11} = \begin{pmatrix} 0.2916 \\ 4.9201 \\ 11.6631 \end{pmatrix} = 11.6631 \begin{pmatrix} 0.025 \\ 0.4219 \\ 1 \end{pmatrix} = 11.6631 X_{12}$$

$$AX_{12} = \begin{pmatrix} 0.2907 \\ 4.9188 \\ 11.6626 \end{pmatrix} = 11.6626 \begin{pmatrix} 0.0249 \\ 0.4218 \\ 1 \end{pmatrix} = 0.6626 X_{13}$$

$$AX_{13} = \begin{pmatrix} 0.2903 \\ 4.9183 \\ 11.6623 \end{pmatrix} = 11.6623 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6623 X_{14}$$

$$AX_{14} = \begin{pmatrix} 0.29 \\ 4.9181 \\ 11.6619 \end{pmatrix} = 11.6619 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6619 X_{15}$$

$$AX_{15} = \begin{pmatrix} 0.29 \\ 4.9181 \\ 11.6619 \end{pmatrix} = 11.6619 \begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix} = 11.6619 X_{16}$$

$\therefore$ The dominant eigen value is 11.6619

The corresponding eigen vector is

$$\begin{pmatrix} 0.0249 \\ 0.4217 \\ 1 \end{pmatrix}.$$

(3) Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot X_2$$

$$AX_2 = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = 7 \cdot X_3$$

$$AX_3 = \begin{pmatrix} 3.5714 \\ 1.8572 \\ 0 \end{pmatrix} = 3.5714\begin{pmatrix} 1 \\ 0.52 \\ 0 \end{pmatrix} = 3.5714\, X_4$$

$$AX_4 = \begin{pmatrix} 4.12 \\ 2.04 \\ 0 \end{pmatrix} = 4.12\begin{pmatrix} 1 \\ 0.4951 \\ 0 \end{pmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{pmatrix} +3.9706 \\ 1.9902 \\ 0 \end{pmatrix} = 3.9706 \begin{pmatrix} 1 \\ 0.5012 \\ 0 \end{pmatrix} = 3.9706 X_6$$

$$AX_6 = \begin{pmatrix} 4.0072 \\ 2.0024 \\ 0 \end{pmatrix} = 4.0072 \begin{pmatrix} 1 \\ 0.4997 \\ 0 \end{pmatrix} = 4.0072 X_7$$

$$AX_7 = \begin{pmatrix} 3.9982 \\ 1.9994 \\ 0 \end{pmatrix} = 3.9982 \begin{pmatrix} 1 \\ 0.5000 \\ 0 \end{pmatrix} = 3.9982 X_8$$

$$AX_8 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4 X_9$$

$$AX_9 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$$

Dominant eigen value is $\lambda = 4$

Corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

Eigen Value of a Matrix by Jacobi Method for Symmetric Matrix

Let $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{ij}}{a_{ii} - a_{jj}}\right)$$

$$D = P^T A P$$

(1) Apply Jacobi process to evaluate the eigen values and eigen vectors of the Matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

Soln

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

The largest non diagonal element is $a_{13} = a_{31} = 1$

$a_{11} = 5$, $a_{33} = 5$

$$\tan 2\theta = \left[\frac{2a_{13}}{a_{11} - a_{33}}\right] = \frac{2}{5-5}$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$P = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi/2 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

I^{st} transformation

$$D = P^T A P$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The eigen values are $6, -2, 4$

corresponding eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(2) Find all the eigen values and eigen vectors of the Matrix

$$\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$ using Jacobi Method.

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Here the largest non diagonal element is $a_{13} = a_{31} = 2$.

$a_{11} = 1, a_{33} = 1$

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0}$$

$$\tan 2\theta = \infty$$

$$2\theta = \pi/2$$

$$\boxed{\theta = \pi/4}$$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_1 = S_1^{-1} A S_1$$

$$= \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt2 & 2 \\ \sqrt2 & 3 & \sqrt2 \\ 2 & \sqrt2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & 0 & -\frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ 1/\sqrt2 & 0 & 1/\sqrt2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

II Transformation

$$a_{12} = a_{21} = 2$$

$$a_{11} = 3 \quad a_{22} = 3$$

$$S_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tan 2\theta = \frac{2a_{12}}{a_{11}-a_{22}} = \frac{2\times 2}{3-3} = \infty$$

$$2\theta = \tan^{-1}\infty$$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

$$S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = S_1^{-1} B_1 S_2$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\therefore$ A is reduced to the diagonal Matrix B_2.

Hence the eigen values of A is 5, 1, −1

$$S = S_1 S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ eigen vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix} \text{ \& } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$