UNIT–IV

BOUNDARY LAYER AND FLOW THROUGH PIPES

Definition of boundary layer – Thickness and classification – Displacement and momentum thickness – Development of laminar and turbulent flows in circular pipes – Major and minor losses of flow in pipes – Pipes in series and in parallel – Pipe network

General Characteristics of External Flow

External flows are defined as the flows immersed in an unbounded fluid. A body immersed in a fluid experiences a resultant force due to the interaction between the body and fluid surroundings. In some cases, the body moves in stationary fluid medium (e.g. motion of an airplane) while in some instances, the fluid passes over a stationary object (e.g. motion of air over a model in a wind tunnel). In any case, one can fix the coordinate system in the body and treat the situation as the flow past a stationary body at a uniform velocity (U), known as upstream/free-stream velocity. However, there are unusual instances where the flow is not uniform. Even, the flow in the vicinity of the object can be unsteady in the case of a steady, uniform upstream flow. For instances, when wind blows over a tall building, different velocities are felt at top and bottom part of the building. But, the unsteadiness and non-uniformity are of minor importance rather the flow characteristic on the surface of the body is more important. The shape of the body (e.g. sharp-tip, blunt or streamline) affects structure of an external flow. For analysis point of view, the bodies are often classified as, two-dimensional objects (infinitely long and constant cross-section), axi-symmetric bodies and three-dimensional objects.

There are a number of interesting phenomena that occur in an external viscous flow past an object. For a given shape of the object, the characteristics of the flow depend very strongly on carious parameters such as size, orientation, speed and fluid properties. The most important dimensionless parameter for a typical external incompressible flow is the Reynolds number $\left(\mathrm{Re} = \dfrac{\rho U l}{\mu}\right)$, which represents the ratio of inertial effects to the viscous effects. In the absence of viscous effects $(\mu = 0)$, the Reynolds number is infinite. In other case, when there are no inertia effects, the Reynolds number is zero. However, the nature of flow pattern in an actual scenario depends strongly on Reynolds number and it falls in these two extremes either $\mathrm{Re} \ll 1$ or $\mathrm{Re} \gg 1$. The typical external flows with air/water are associated moderately sized objects with certain characteristics length $(0.01\mathrm{m} < l < 10\mathrm{m})$ and free stream velocity $/s < U < 100\,/s)$ that results Reynolds number in the range $10 < \mathrm{Re} < 10^9$. So, as a rule of thumb, the flows with Re 1 , are dominated by

viscous effects and inertia effects become predominant when $\mathrm{Re} > 100$. Hence, the most familiar external flows are dominated by inertia. So, the objective of this section is to quantify the behavior of viscous, incompressible fluids in external flow.

Let us discuss few important features in an external flow past an airfoil (Fig. 5.7.1) where the flow is dominated by inertial effects. Some of the important features are highlighted below;

- The free stream flow divides at the stagnation point.
- The fluid at the body takes the velocity of the body (no-slip condition).
- A boundary layer is formed at the upper and lower surface of the airfoil.
- The flow in the boundary layer is initially laminar and the transition to turbulence takes place at downstream of the stagnation point, depending on the free stream conditions.
- The turbulent boundary layer grows more rapidly than the laminar layer, thus thickening the boundary layer on the body surface. So, the flow experiences a thicker body compared to original one.
- In the region of increasing pressure (adverse pressure gradient), the flow separation may occur. The fluid inside the boundary layer forms a viscous wake behind the separated points.

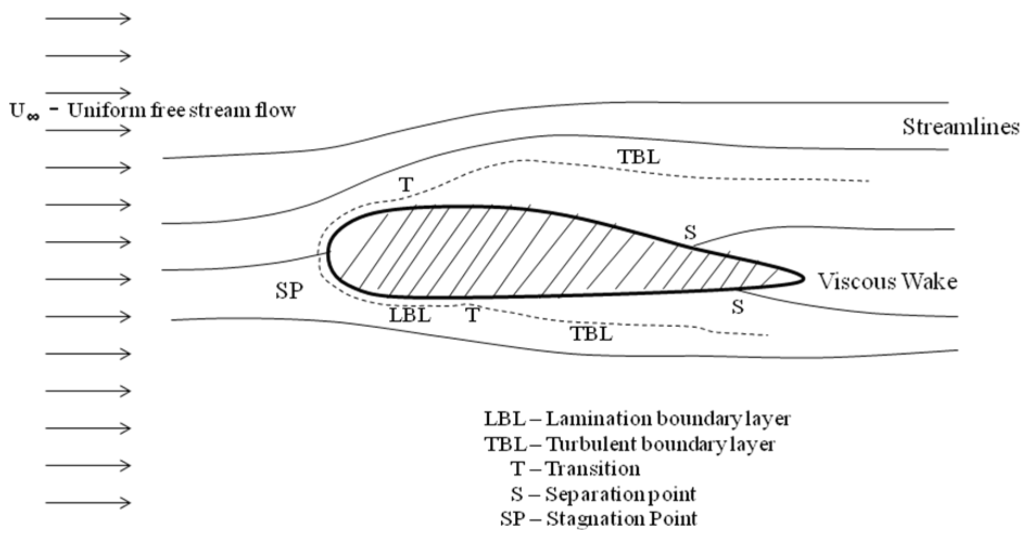

Fig. 5.7.1: Important features in an external flow.

Boundary Layer Characteristics

The concept of boundary layer was first introduced by a German scientist, Ludwig Prandtl, in the year 1904. Although, the complete descriptions of motion of a viscous fluid were known through Navier-Stokes equations, the mathematical difficulties in solving these equations prohibited the theoretical analysis of viscous flow. Prandtl suggested that the viscous flows can be analyzed by dividing the flow into two regions; one close to the solid boundaries and other covering the rest of the flow. Boundary layer is the regions close to the solid boundary where the effects of viscosity are experienced by the flow. In the regions outside the boundary layer, the effect of viscosity is negligible and the fluid is treated as inviscid. So, the boundary layer is a buffer region between the wall below and the inviscid free-stream above. This approach allows the complete solution of viscous fluid flows which would have been impossible through Navier-Stokes equation. The qualitative picture of the boundary-layer growth over a flat plate is shown in Fig. 5.7.2.

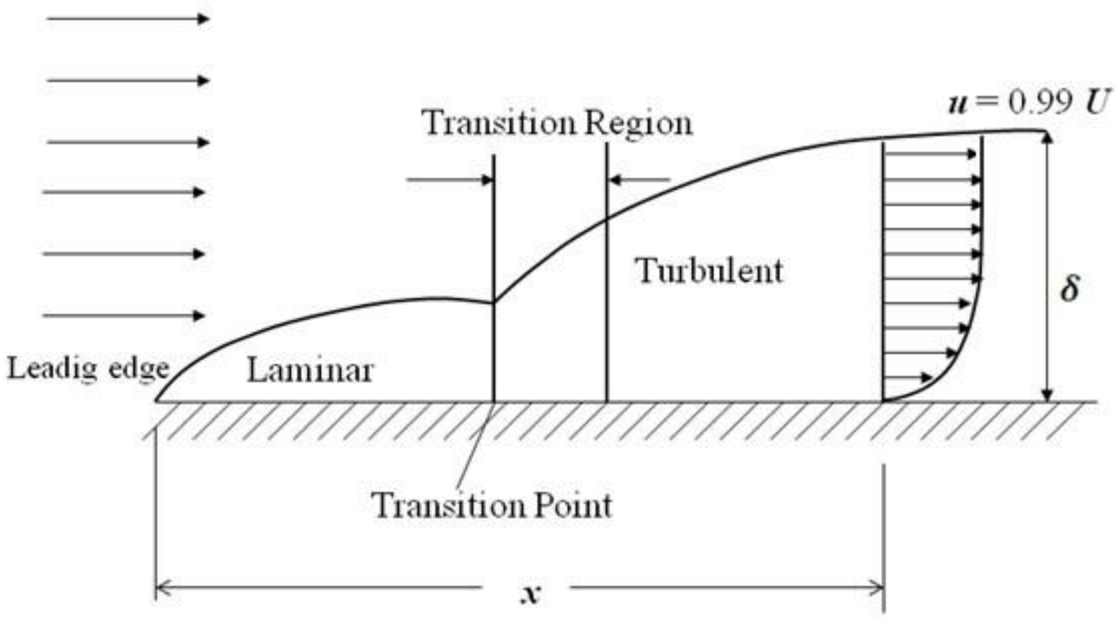

Fig. 5.7.2: Representation of boundary layer on a flat plate.

A laminar boundary layer is initiated at the leading edge of the plate for a short distance and extends to downstream. The transition occurs over a region, after certain length in the downstream followed by fully turbulent boundary layers. For common calculation purposes, the transition is usually considered to occur at a distance where the Reynolds number is about 500,000. With air at standard conditions, moving at a velocity of 30m/s, the transition is expected to occur at a distance of about 250mm. A typical boundary layer flow is characterized by certain parameters as given below;

Boundary layer thickness (δ) : It is known that no-slip conditions have to be satisfied at the solid surface: the fluid must attain the zero velocity at the wall. Subsequently, above the wall, the effect of viscosity tends to reduce and the fluid within this layer will try to approach the free stream velocity. Thus, there is a velocity gradient that develops within the fluid layers inside the small regions near to solid surface. The *boundary layer thickness* is defined as the distance from the surface to a point where the velocity is reaches 99% of the free stream velocity. Thus, the velocity profile merges smoothly and asymptotically into the free stream as shown in Fig. 5.7.3(a).

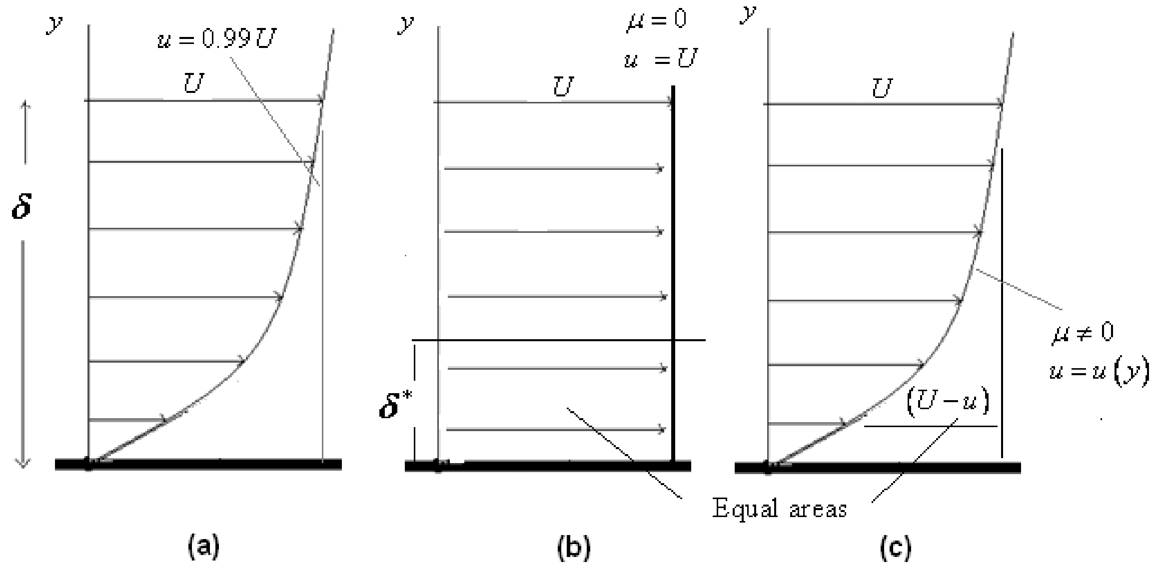

Fig. 5.7.3: (a) Boundary layer thickness; (b) Free stream flow (no viscosity); (c) Concepts of displacement thickness.

<u>Displacement thickness</u> $\left(\delta^*\right)$: The effect of viscosity in the boundary layer is to retard the flow. So, the mass flow rate adjacent to the solid surface is less than the mass flow rate that would pass through the same region in the absence of boundary layer. In the absence of viscous forces, the velocity in the vicinity of sold surface would be U as shown in Fig. 5.7.3(b). The decrease in the mass flow rate due to the influence of viscous forces is $\int_0^\infty \rho\left(U-u\right)b\,dy$, where b is the width of the surface in the direction perpendicular to the flow. So, the *displacement thickness* is the distance by which the solid boundary would displace in a frictionless flow (Fig. 5.7.3-b) to give rise to same mass flow rate deficit as exists in the boundary layer (Fig. 5.7.3-c). The mass flow rate deficiency by displacing the solid boundary by δ^* will be $\rho U \delta^* b$. In an incompressible flow, equating these two terms, the expression for δ^* is obtained.

$$\begin{aligned} &\rho U \delta^* b = \int_0^\infty \rho\left(U-u\right)b\,dy \\ &\Rightarrow \delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy \approx \int_0^\delta \left(1-\frac{u}{U}\right)dy \end{aligned} \tag{5.7.1}$$

<u>Momentum thickness</u> $\left(\theta^*\right)$: The flow retardation in the boundary layer also results the reduction in momentum flux as compared to the inviscid flow. The momentum thickness is defined as the thickness of a layer of fluid with velocity U, for which the momentum flux is equal to the deficit of momentum flux through the boundary layer. So, the expression for θ^* in an incompressible flow can be written as follow;

$$\begin{aligned} &\rho U^2 \theta^* = \int_0^\infty \rho u\left(U-u\right)dy \\ &\Rightarrow \theta^* = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy \approx \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy \end{aligned} \tag{5.7.2}$$

The displacement/momentum thickness has the following physical implications;

- The displacement thickness represents the amount of distance that thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flow rate properties as the actual flow.
- It indicates the outward displacement of the streamlines caused by the viscous effects on the plate.

- The flow conditions in the boundary layer can be simulated by adding the displacement thickness to the actual wall thickness and thus treating the flow over a thickened body as in the case of inviscid flow.
- Both δ^* and θ^* are the integral thicknesses and the integrant vanishes in the free stream. So, it is relatively easier to evaluate δ^* and θ^* as compared to δ.

The boundary layer concept is based on the fact that the boundary layer is thin. For a flat plate, the flow at any location x along the plate, the boundary layer relations $\left(\delta \ll x;\ \delta^* \ll x \text{ and } \theta^* \ll x\right)$ are true except for the leading edge. The velocity profile merges into the local free stream velocity asymptotically. The pressure variation across the boundary layer is negligible i.e. same free stream pressure is impressed on the boundary layer. Considering these aspects, an approximate analysis can be made with the following assumptions within the boundary layer.

$$\text{At } y = \delta \Rightarrow u \to U$$
$$\text{At } y = \delta \Rightarrow \left(\partial u / \partial y\right) \to 0 \qquad (5.7.3)$$
$$\text{Within the boundary layer, } v \ll U$$

Module 5 : Lecture 8
VISCOUS INCOMPRESSIBLE FLOW
(External Flow – Part II)

Boundary Layer Equations

There are two general flow situations in which the viscous terms in the Navier-Stokes equations can be neglected. The first one refers to high Reynolds number flow region where the net viscous forces are negligible compared to inertial and/or pressure forces, thus known as *inviscid flow region*. In the other cases, there is no vorticity (irrotational flow) in the flow field and they are described through potential flow theory. In either case, the removal of viscous terms in the Navier-Stokes equation yields Euler equation. When there is a viscous flow over a stationary solid wall, then it must attain zero velocity at the wall leading to non-zero viscous stress. The Euler's equation has the inability to specify no-slip condition at the wall that leads to un-realistic physical situations. The gap between these two equations is overcome through *boundary layer approximation developed by Ludwig Prandtl* (1875-1953). The idea is to divide the flow into two regions: outer inviscid/irrotational flow region and *boundary laye*r region. It is a very thin inner region near to the solid wall where the vorticity/irrotationality cannot be ignored. The flow field solution of the inner region is obtained through boundary layer equations and it has certain assumptions as given below;

- The thickness of the boundary layer (δ) is very small. For a given fluid and plate, if the Reynolds number is high, then at any given location (x) on the plate, the boundary layer becomes thinner as shown in Fig. 5.8.1(a).
- Within the boundary layer (Fig. 5.8.1-b), the component of velocity normal to the wall is very small as compared to tangential velocity $(v \ll u)$.
- There is no change in pressure across the boundary layer i.e. pressure varies only in the *x*-direction.

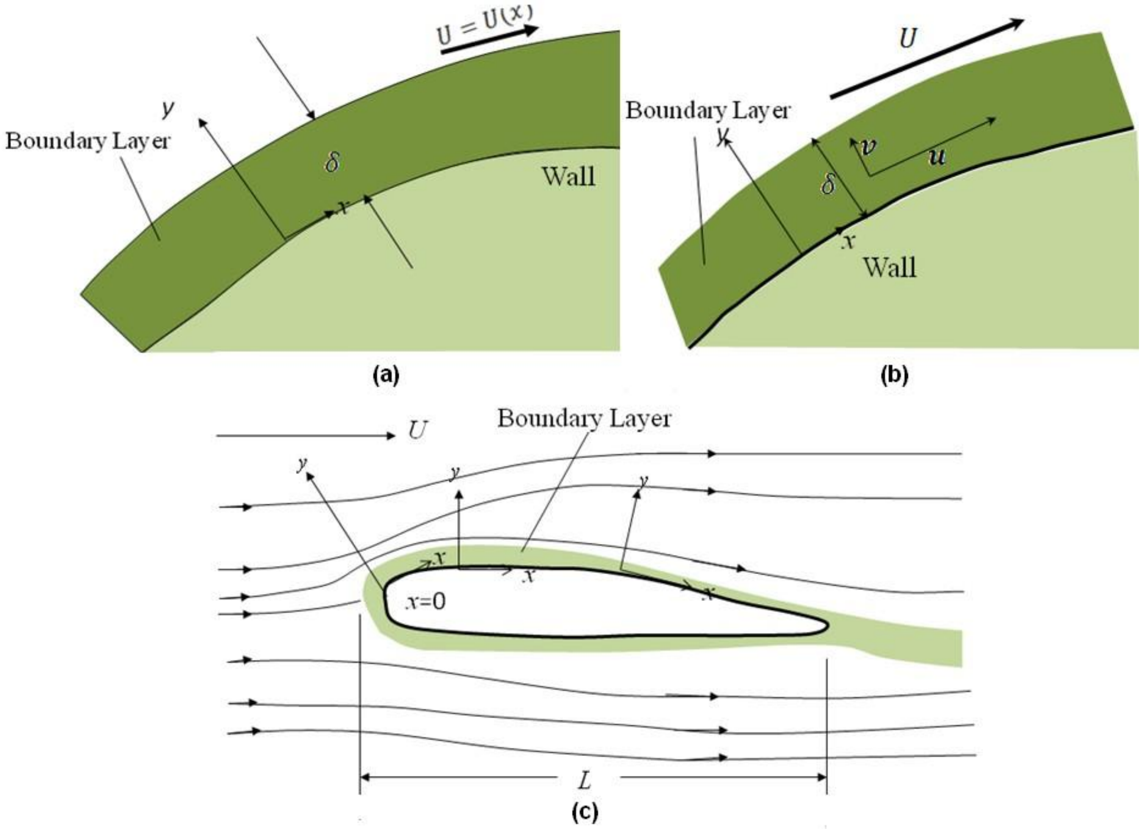

Fig. 5.8.1: Boundary layer representation: (a) Thickness of boundary layer; (b) Velocity components within the boundary layer; (c) Coordinate system used for analysis within the boundary layer.

After having some physical insight into the boundary layer flow, let us generate the boundary layer equations for a steady, laminar and two-dimensional flow in *x*-*y* plane as shown in Fig. 5.8.1(c). This methodology can be extended to axi-symmetric/three-dimensional boundary layer with any coordinate system. Within the boundary-layer as shown in Fig. 5.8.1(c), a coordinate system is adopted in which *x* is parallel to the wall everywhere and *y* is the direction normal to the wall. The location $x = 0$ refers to stagnation point on the body where the free stream flow comes to rest. Now, take certain length scale (L) for distances in the stream-wise direction (x) so that the derivatives of velocity and pressure can be obtained. Within the boundary layer, the choice of this length scale (L) is too large compared to the boundary layer thickness (δ). So the scale *L* is not a proper choice for *y*-direction. Moreover, it is difficult to obtain the derivatives with respect to *y*. So, it is more appropriate to use a length scale of δ for the direction normal to the stream-wise direction. The characteristics velocity $U = U(x)$ is the velocity parallel to the wall at a location just above the boundary layer and p_∞ is the free stream pressure. Now, let us perform *order of magnitude* analysis within the boundary layer;

$$u \sim U; \quad (p - p_\infty) \sim \rho U^2; \quad \frac{\partial}{\partial x} \sim \frac{1}{L}; \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta} \tag{5.8.1}$$

Now, apply Eq. (5.8.1) in continuity equation to obtain order of magnitude in *y*-component of velocity.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{U}{L} \sim \frac{v}{\delta} \quad \Rightarrow v \sim \frac{U\delta}{L} \tag{5.8.2}$$

Consider the momentum equation in the *x* and *y* directions;

$$\begin{aligned} x-\text{momentum}: u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\left(\frac{dP}{dx}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ y-\text{momentum}: u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\left(\frac{\partial P}{\partial y}\right) + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned} \tag{5.8.3}$$

Here, $\nu\left(=\frac{\mu}{\rho}\right)$ is the kinematic viscosity. Let us define non-dimensional variables within the boundary layer as follows:

$$x^* = \frac{x}{L}; \; y^* = \frac{y}{\delta}; \; u^* = \frac{u}{U}; \; v^* = \frac{vL}{U\delta}; \; p^* = \frac{p - p_\infty}{\rho U^2} \tag{5.8.4}$$

First, apply Eq. (5.8.4) in *y*-momentum equation, multiply each term by $L^2/(U^2\delta)$ and after simplification, one can obtain the non-dimensional form of *y* – momentum equation.

$$\begin{aligned} u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} &= -\left(\frac{L}{\delta}\right)^2\frac{\partial p^*}{\partial y^*} + \left(\frac{\nu}{UL}\right)\frac{\partial^2 v^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}} \\ \text{or, } u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + \left(\frac{L}{\delta}\right)^2\frac{\partial p^*}{\partial y^*} &= \left(\frac{1}{\text{Re}}\right)\frac{\partial^2 v^*}{\partial x^{*2}} + \left(\frac{1}{\text{Re}}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}} \end{aligned} \tag{5.8.5}$$

For boundary layer flows, the Reynolds number is considered as very high which means the second and third terms in the RHS of Eq. (5.8.5) can be neglected. Further, the pressure gradient term is much higher than the convective terms in the LHS of Eq. (5.8.5), because $L \gg \delta$. So, the non-dimensional *y*-momentum equation reduces to,

$$\frac{\partial p^*}{\partial y^*} \cong 0 \quad \Rightarrow \frac{\partial p}{\partial y} = 0 \tag{5.8.6}$$

It means the pressure across the boundary layer (y-direction) is nearly constant i.e. negligible change in pressure in the direction normal to the wall (Fig. 5.8.2-a). This leads to the fact that the streamlines in the thin boundary layer region have negligible curvature when observed at the scale of δ. However, the pressure may vary along the wall (x-direction). Thus, y-momentum equation analysis suggests the fact that pressure across the boundary layer is same as that of inviscid outer flow region. Hence, one can apply Bernoulli equation to the outer flow region and obtain the pressure variation along x-direction where both p and U are functions of x only (Fig. 5.8.2-b).

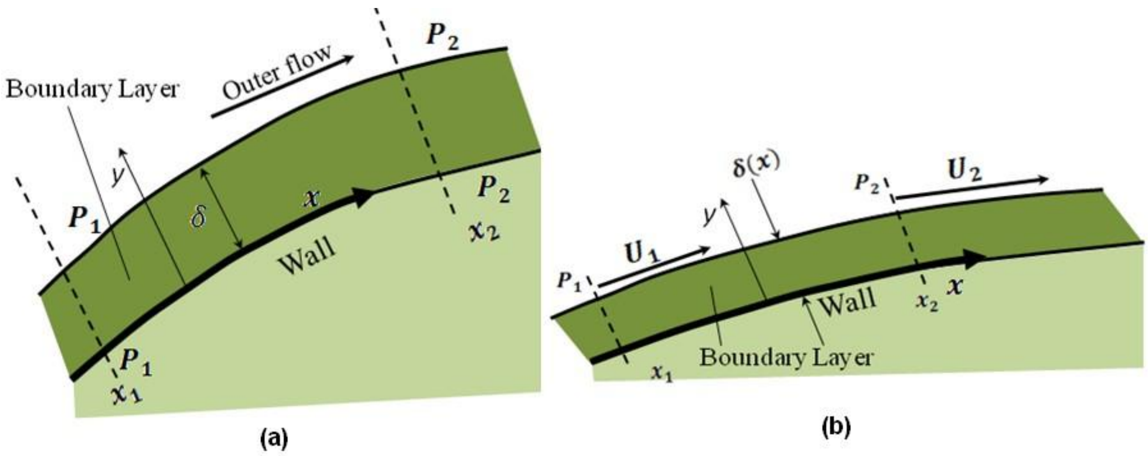

Fig. 5.8.2: Variation of pressure within the boundary layer: (a) Normal to the wall; (b) Along the wall.

$$\frac{p}{\rho}+\frac{1}{2}U^2=\text{constant} \quad \Rightarrow \frac{1}{\rho}\frac{dp}{dx}=-U\frac{dU}{dx} \tag{5.8.7}$$

Next, apply Eq. (5.8.4) in x – momentum equation, multiply each term by $\left(L/U^2\right)$ and after simplification, one can obtain the non-dimensional form of x– momentum equation.

$$\begin{aligned} & u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{dp^*}{dx^*}+\left(\frac{\nu}{UL}\right)\frac{\partial^2 u^*}{\partial x^{*2}}+\left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}} \\ \text{or, } & u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=\frac{\partial p^*}{\partial x^*}+\left(\frac{1}{\mathrm{Re}}\right)\frac{\partial^2 u^*}{\partial x^{*2}}+\left(\frac{1}{\mathrm{Re}}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}} \end{aligned} \tag{5.8.8}$$

It may be observed that all the terms in the LHS and first term in the RHS of Eq. (5.8.8) are of the order unity. The second term of RHS can be neglected because the Reynolds number is considered as very high. The last term of Eq. (5.8.8) is equivalent to inertia term and thus it has to be the order of one.

$$\left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \sim 1 \quad \Rightarrow \frac{U^2}{L} \sim \nu\frac{U}{\delta^2} \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}} \qquad (5.8.9)$$

Eq. (5.8.9) clearly shows that the convective flux terms are of same order of magnitudes of viscous diffusive terms. Now, neglecting the necessary terms and with suitable approximations, the equations for a steady, incompressible and laminar boundary flow can be obtained from Eqs (5.8.2 & 5.8.3). They are written in terms of physical variables in *x*-*y* plane as follows;

$$\begin{aligned}
&\text{Continuity}: \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \\
&x-\text{momentum}: u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \\
&y-\text{momentum}: \frac{\partial p}{\partial y}=0
\end{aligned} \qquad (5.8.10)$$

Solution Procedure for Boundary Layer

Mathematically, a full Navier-Stokes equation is elliptic in space which means that the boundary conditions are required in the entire flow domain and the information is passed in all directions, both upstream and downstream. However, with necessary boundary layer approximations, the *x* – momentum equation is parabolic in nature which means the boundary conditions are required only three sides of flow domain (Fig. 5.8.3-a). So, the stepwise procedure is outlined here.

- Solve the outer flow with inviscid/irrotational assumptions using Euler's equation and obtain the velocity field as $U(x)$. Since the boundary layer is very thin, it does not affect the outer flow solution.
- With some known starting profile $\left(x=x_s \Rightarrow u=u_s(y)\right)$, solve the Eq. (5.8.10) with no-slip conditions at the wall $(y=0 \Rightarrow u=v=0)$ and known outer flow condition at the edge of the boundary layer $\left(y\to\infty \Rightarrow u=U(x)\right)$
- After solving Eq. (5.8.10), one can obtain all the boundary layer parameters such as displacement and momentum thickness.

Even though the boundary layer equations (Eq. 5.8.10) and the boundary conditions seem to be simple, but no analytical solution has been obtained so far. It was first solved numerically in the year 1908 by *Blasius*, for a simple flat plate. Nowadays, one can solve these equations with highly developed computer tools. It will be discussed in the subsequent section.

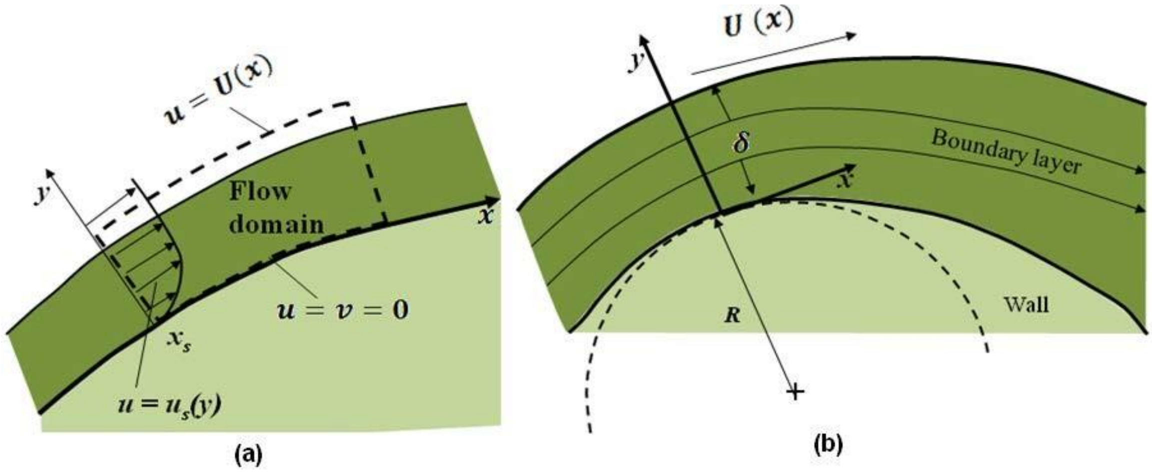

Fig. 5.8.3: Boundary layer calculations: (a) Initial condition and flow domain; (b) Effect of centrifugal force.

Limitations of Boundary Calculations

- The boundary layer approximation fails if the Reynolds number is not very large. Referring to Eq. (5.8.9), one can interpret $(\delta/L) \square\, 0.001 \Rightarrow \mathrm{Re}_L \square\, 10000$.
- The assumption of zero-pressure gradient does not hold good if the wall curvature is of similar magnitude as of δ because of centrifugal acceleration (Fig. 5.8.3-b).
- If the Reynolds number is too high $\mathrm{Re}_L \square\, 10^5$, then the boundary layer does not remain laminar rather the flow becomes transitional or turbulent. Subsequently, if the flow separation occurs due to adverse pressure gradient, then the parabolic nature of boundary layer equations is lost due to flow reversal.

Module 5 : Lecture 9
VISCOUS INCOMPRESSIBLE FLOW
(External Flow – Part III)

Laminar Boundary Layer on a Flat Plate

Consider a uniform free stream of speed (U) that flows parallel to an infinitesimally thin semi-infinite flat plate as shown in Fig. 5.9.1(a). A coordinate system can be defined such that the flow begins at *leading edge* of the plate which is considered as the origin of the plate. Since the flow is symmetric about *x*-axis, only the upper half of the flow can be considered. The following assumptions may be made in the discussions;

- The nature of the flow is steady, incompressible and two-dimensional.
- The Reynolds number is high enough that the boundary layer approximation is reasonable.
- The boundary layer remains laminar over the entire flow domain.

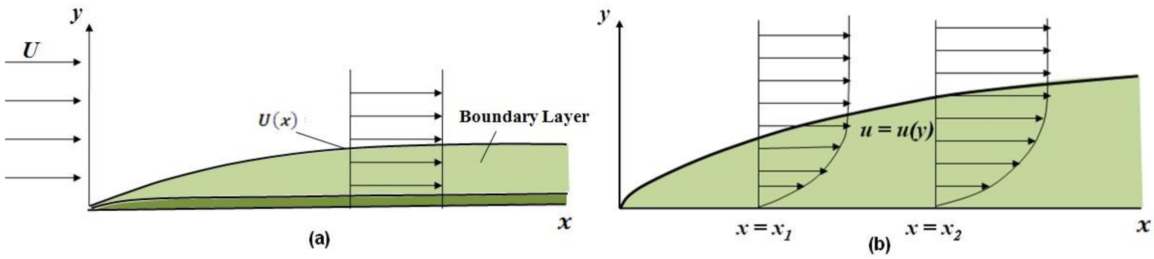

Fig. 5.9.1: Boundary layer on a flat plate: (a) Outer inviscid flow and thin boundary layer; (b) Similarity behavior of boundary layer at any x-location.

The outer flow is considered without the boundary layer and in this case, U is a constant so that $U\dfrac{dU}{dx}=0$. Referring to Fig. 5.9.1, the boundary layer equations and its boundary conditions can be written as follows;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0; \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}; \quad \frac{\partial p}{\partial y}=0 \tag{5.9.1}$$

Boundary conditions:

$$\begin{aligned} &u=0 \text{ at } y=0 \text{ and } \ u=U; \frac{du}{dy}=0 \text{ at } \ y\to\infty \\ &v=0 \text{ at } y=0 \ \text{ and } u=U \ \text{ for all } y \ \text{ at } \ x=0 \end{aligned} \tag{5.9.2}$$

No analytical solution is available till date for the above boundary layer equations. However, this equation was solved first by numerically in the year 1908 by *P.R.Heinrich Blasius* and commonly known as *Blasius solution for laminar boundary layer over a flat plate*. The key for the solution is the assumption of *similarity* which means there is no characteristics length scale in the geometry of the problem. Physically, it is the case for the same flow patterns for an infinitely long flat plate regardless of any close-up view (Fig. 5.9.1-b). So, mathematically a *similarity variable* (η) can be defined that combines the independent variables *x and y* into a non-dimensional independent variable. In accordance with the similarity law, the velocity profile is represented in the functional form;

$$\frac{u}{U}=F(x,y,\nu,U)=F(\eta) \quad \Rightarrow u=U\{F(\eta)\} \tag{5.9.3}$$

With the *order of magnitude* analysis, the thickness of the boundary layer is interpreted as $\delta \sim \sqrt{\frac{\nu x}{U}}$. Based on this analogy, *Blasius* set the non-dimensional similarity variable in the following functional form;

$$\eta = y\sqrt{\frac{U}{\nu x}} \quad \Rightarrow \frac{d\eta}{dx}=-\frac{\eta}{2x} \text{ and } \frac{d\eta}{dy}=\sqrt{\frac{U}{\nu x}} \tag{5.9.4}$$

Now, let us introduce the stream function (ψ) for the two-dimensional flow.

$$u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x} \tag{5.9.5}$$

The stream function can be obtained through integration of Eq. (5.9.5) and using the results of Eqs (5.9.3 & 5.9.4).

$$\psi=\int u\,dy=\int U\,F(\eta)\sqrt{\frac{\nu x}{U}}d\eta=\sqrt{U\nu x}\int F(\eta)d\eta=\sqrt{U\nu x}\,f(\eta) \tag{5.9.6}$$

Again, differentiate Eq. (5.9.6) with respect to *y* and use Eq. (5.9.4) to obtain the *x*-component of velocity profile within the boundary layer as function η (Fig. 5.9.2).

$$u=\frac{\partial\psi}{\partial y}=\sqrt{U\nu x}\left(\frac{\partial f}{\partial\eta}\right)\left(\frac{\partial\eta}{\partial y}\right)=U f'(\eta) \quad \Rightarrow \frac{u}{U}=f'(\eta) \tag{5.9.7}$$

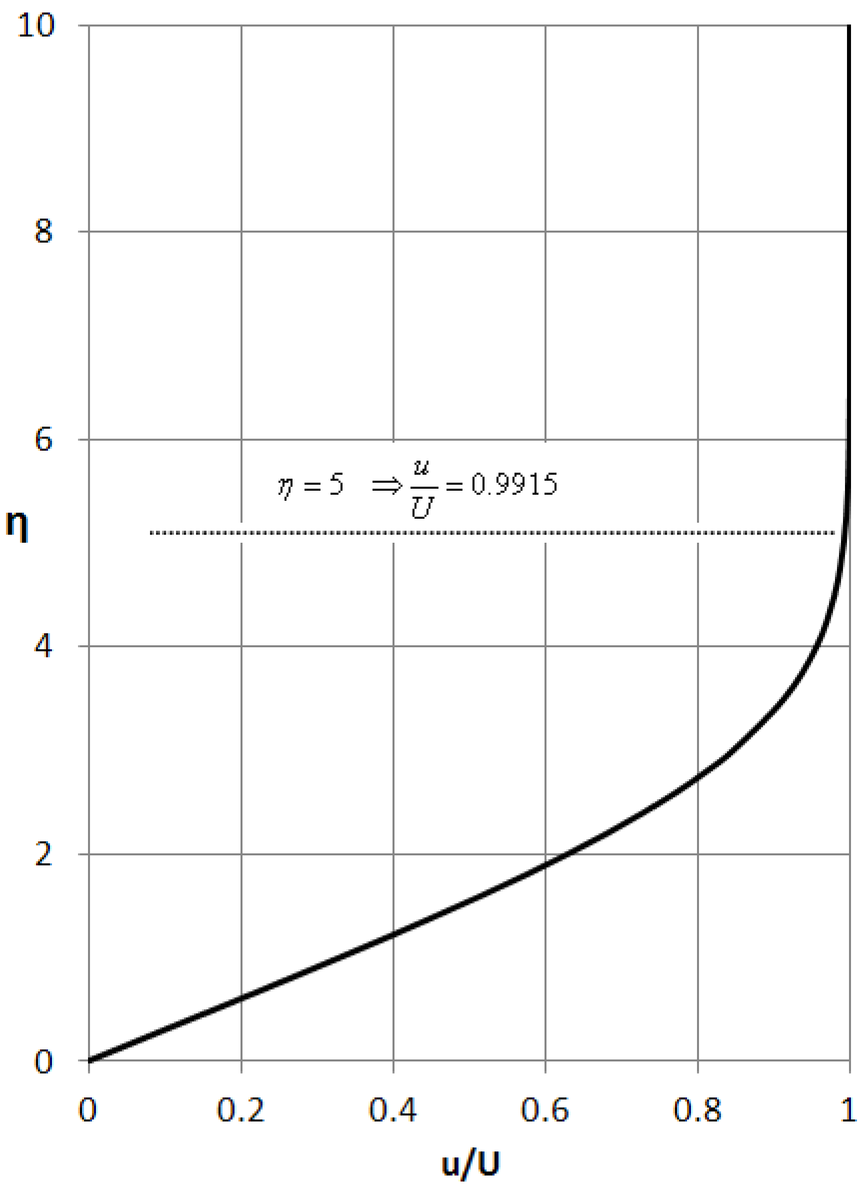

Fig. 5.9.2: Blasius profile for a laminar boundary layer over a flat plate.

The *y*-component of velocity profile can be obtained by differentiating stream function with respect to *x* and substituting the results from Eq. (5.9.4 & 5.9.6).

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{U\nu x}\, f(\eta)\right] = -\left[\frac{1}{2}\sqrt{\frac{U\nu}{x}}\, f(\eta) + \sqrt{U\nu x}\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

$$= -\left[\frac{1}{2}\sqrt{\frac{U\nu}{x}}\, f(\eta) - \frac{1}{2}\sqrt{\frac{U\nu}{x}}\,\eta f'(\eta)\right] = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left[\eta f'(\eta) - f(\eta)\right] \tag{5.9.8}$$

Now, let us calculate each term of Eq. (5.9.1) from the velocity components obtained from Eqs (5.9.7 & 5.9.8).

$$\frac{\partial u}{\partial x} = -\frac{U}{2x}\eta f''(\eta);\quad \frac{\partial u}{\partial y} = U f''(\eta)\sqrt{\frac{U}{\nu x}};\quad \frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x} f'''(\eta)$$
$$u = U f'(\eta);\ v = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left[\eta f'(\eta) - f(\eta)\right] \tag{5.9.9}$$

Substitute each term of Eq. (5.9.9) in Eq. (5.9.1) and after simplification, the boundary layer equation reduces to *Blasius* equation expressed in terms of similarity variable.

$$2 f''' + f\, f'' = 0 \quad \Rightarrow f''' + \frac{1}{2} f f'' = 0 \tag{5.9.10}$$

Table 5.9.1: Solution of *Blasius* laminar flat plate boundary layer in similarity variables

$\eta = y\sqrt{\frac{U}{\nu x}}$	$f(\eta)$	$f'(\eta) = \frac{u}{U}$	$f''(\eta)$
0	0	0	0.3321
1	0.1656	0.3298	0.3230
2	0.650	0.6298	0.2668
3	1.3968	0.8460	0.1614
4	2.3057	0.9555	0.0642
5	3.2833	0.9915	0.0159
6	4.2796	0.9990	0.0024
7	5.2792	0.9999	0.0002
8	6.2792	1.0	0
9	7.2792	1.0	0
10	8.2792	1.0	0

In certain cases, one can define $\eta = y\sqrt{\frac{U}{2\nu x}}$ and $\psi = \sqrt{2U\nu x}\, f(\eta)$ for which Eq. (5.9.10) takes the following form;

$$f''' + f\, f'' = 0 \tag{5.9.11}$$

The Blasius equation is a third-order non-linear ordinary differential equation for which the boundary conditions can be set using Eq. (5.9.2).

$$\begin{aligned} &y = 0;\, u = 0 && \Rightarrow \eta = 0,\, f' = 0 \\ &y = 0;\, v = 0 && \Rightarrow \eta = 0, f = f' = 0 \\ &y \to \infty;\, u = U && \Rightarrow \eta = \infty,\; f' = 1 \end{aligned} \tag{5.9.12}$$

The popular *Runge-Kutta numerical technique* can be applied for Eqs (5.9.11 & 5.9.12) to obtain the similarity solution in terms of η and some of the values are given in the Table 5.9.1.

Estimation of Boundary Layer Parameters

Boundary layer thickness (δ) : It is defined as the distance away from the wall at which the velocity component parallel to the wall is 99% of the fluid speed outside the boundary layer. From Table 5.9.1, it is seen that $f'(\eta) = \frac{u}{U} = 0.99$ at $\eta = 5$. So, replacing $y = \delta$ in Eq. (5.9.4), one can obtain the following expression for boundary layer thickness;

$$\delta = \frac{5}{\sqrt{\frac{U}{\nu x}}} = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = \frac{5x}{\sqrt{\mathrm{Re}_x}} \quad \Rightarrow \frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}} \quad \text{and} \quad \delta \propto x^{\frac{1}{2}} \tag{5.9.13}$$

At this point, the transverse velocity can be calculated from Eq. (5.9.8).

For $\eta = 5$, $f(\eta) = 3.2833$ and $f'(\eta) = 0.9915$

$$\frac{v}{U} = \frac{1}{2\sqrt{\frac{Ux}{\nu}}}\left[\eta f'(\eta) - f(\eta)\right] = \frac{1}{2\sqrt{\mathrm{Re}_x}}\left[\eta f'(\eta) - f(\eta)\right] \quad \Rightarrow \frac{v}{U} = \frac{0.84}{\sqrt{\mathrm{Re}_x}} \tag{5.9.14}$$

Displacement thickness $\left(\delta^*\right)$: It is the distance that a streamline just outside of the boundary layer is deflected away from the wall due to the effect of the boundary layer. Mathematically, it can be represented in terms of transformed variable using Eq. (5.9.4).

$$\delta^* = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy = \sqrt{\frac{\nu x}{U}}\int_0^{\eta}\left[1-f'(\eta)\right]d\eta = = \sqrt{\frac{\nu x}{U}}\left[\eta - f(\eta)\right] \qquad (5.9.15)$$

It may be seen from the Table 5.9.1 that for all values of $\eta > 5$, the functional value of Eq. (5.9.15) is always a constant quantity i.e. $\left[\eta - f(\eta)\right] = 1.72$. So, Eq. (5.9.15) can be simplified in terms of Reynolds number;

$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}} \qquad (5.9.16)$$

Momentum thickness $\left(\theta^*\right)$: It is defined as the loss of momentum flux per unit width divided by ρU^2 due to the presence of the growing boundary layer. Mathematically, it can be represented in terms of transformed variable using Eq. (5.9.4).

$$\theta^* = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy = \sqrt{\frac{\nu x}{U}}\int_0^{\eta} f'(\eta)\left[1-f'(\eta)\right]d\eta \qquad (5.9.17)$$

This integration is carried out numerically from $\eta = 0$ to any arbitrary point $\eta > 5$ and the results give rise to the following relation;

$$\frac{\theta^*}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad (5.9.18)$$

Comparing the Eqs (5.9.13, 5.9.16 & 5.9.18), it is seen that the all are inversely proportional to the square root of Reynolds number except the difference in magnitude. The value of δ^* is about 34% of δ while θ^* turns out to be approximately 13% of δ at any *x*-location (Fig. 5.9.3).

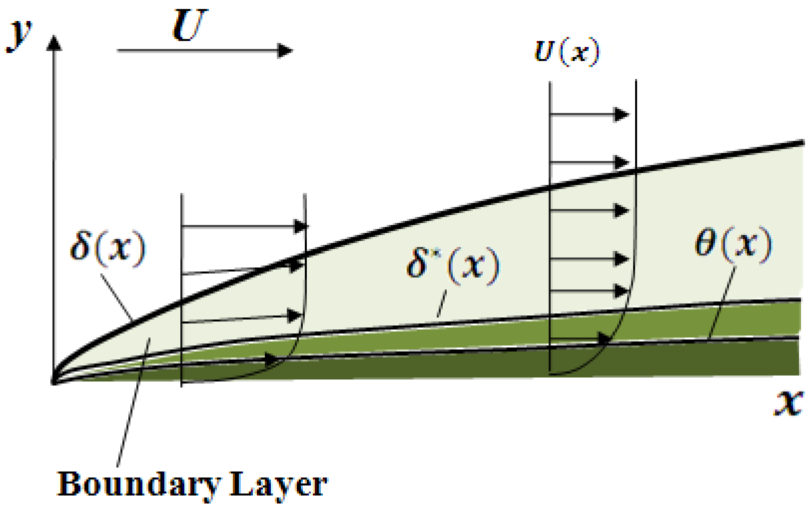

Fig. 5.9.3: Boundary layer thickness, displacement thickness and momentum thickness for a flat plate.

In order to correlate the data for variety of boundary layers under different conditions, a *dimensionless profile shape factor* is often defined as the ratio of displacement thickness to momentum thickness. For a flat plate laminar boundary layer,

$$H = \frac{\delta_*}{\theta^*} = 2.59 \qquad (5.9.19)$$

<u>Skin friction coefficient</u> (c_f): Analogous to friction factor in a duct/pipe flows, a non-dimensional parameter is defined for boundary layer flow as the *skin-friction coefficient* (Fig. 5.9.4). It relates the shear stress at the wall (τ_w) to the free stream dynamic pressure.

$$c_f = \frac{\tau_w}{(1/2)\rho U^2} = \frac{\mu/\partial y)_{y=0}}{(1/2)\rho U^2} = \frac{\mu\left[U f''(\eta)\,(\partial\eta/\partial y)\right]_{y=0}}{(1/2)\rho U^2}$$
$$= \frac{\mu U\sqrt{U/\nu x}}{(1/2)\rho U^2} f''(0) = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \qquad (5.9.20)$$

From Eqs (5.9.18 & 5.9.20), it is observed that the non-dimensional momentum thickness is identical to the skin friction coefficient. Further, the wall shear stress can be estimated from Eq. 5.9.20.

$$\tau_w = \frac{0.664}{2}\rho U^2 \sqrt{\frac{\mu}{\rho U x}} = 0.332 U^{\frac{3}{2}} \sqrt{\frac{\rho\mu}{x}} \qquad (5.9.21)$$

It may be noted here that shear stress decreases with increase in the value of x because of the increase in the boundary layer thickness and decrease in velocity gradient at the wall along the direction of x. Also, τ_w varies directly with $U^{\frac{3}{2}}$ not as U which is the case for a fully-developed laminar pipe flow.

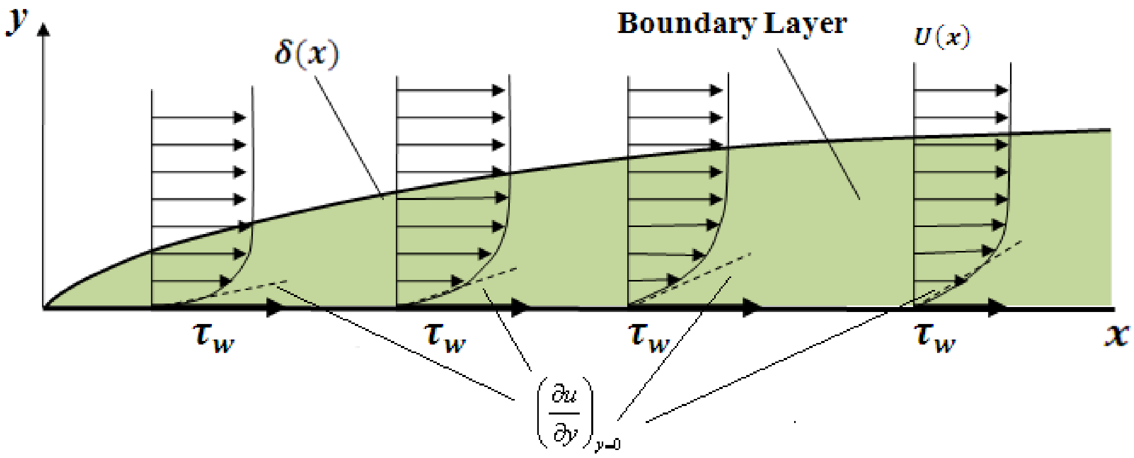

Fig. 5.9.4: Decay of wall shear stress due to decrease in slope at the wall.

<u>Drag coefficient</u> (c_d) : The effect of skin friction/wall shear stress is to retard the free stream flow. It is quantified by skin friction coefficient at a particular *x*-location on the flat plate and is expressed by Eq. (5.9.20). When this parameter is integrated over the entire length of the plate, then total drag coefficient is obtained (Fig. 5.9.5). In terms of free stream parameter and wall shear stress, c_d is quantified as follows;

$$c_d = \frac{D}{(1/2)\rho U^2 A} = \frac{1}{L}\int_0^L c_f\, dx = \frac{0.664}{L}\sqrt{U/\nu L}\int_0^L x^{-1/2}\, dx = \frac{1.328}{\sqrt{\mathrm{Re}_L}} \qquad (5.9.22)$$

Comparing Eqs (5.9.18 & 5.9.22), it is seen that skin friction drag coefficient for a flat plate is directly proportional to the values of θ^* evaluated at the trailing edge of the plate.

$$\left[\theta^*\right]_{x=L} = \frac{0.664\, L}{\sqrt{\mathrm{Re}_L}};\ c_d = \frac{1.328}{\sqrt{\mathrm{Re}_L}} = \frac{2\left[\theta^*\right]_{x=L}}{L} \qquad (5.9.23)$$

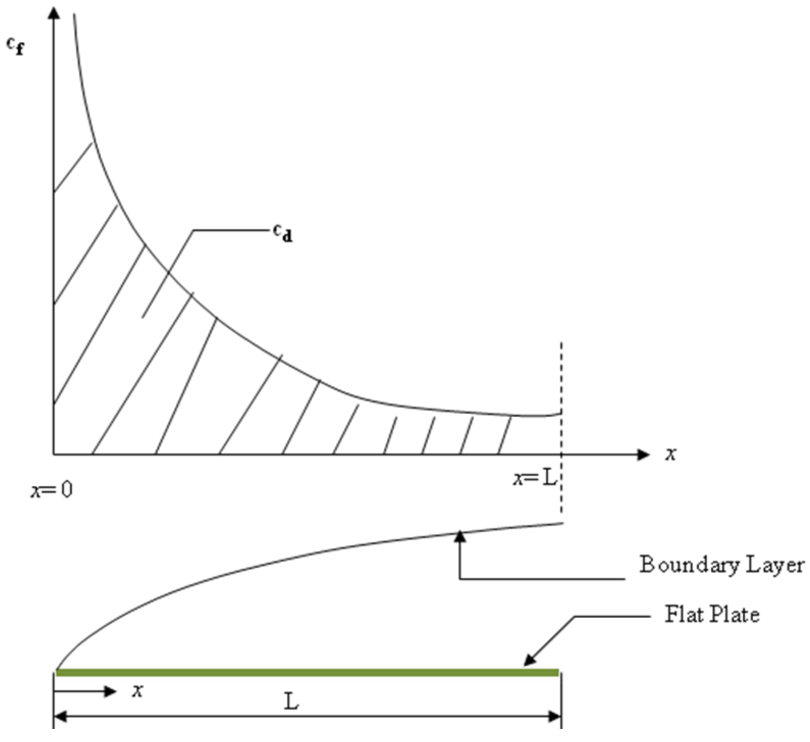

Fig. 5.9.5: Variation of skin friction coefficient along the length of a flat plate.

Module 5 : Lecture 10
VISCOUS INCOMPRESSIBLE FLOW
(External Flow – Part IV)

Momentum Integral Boundary Layer Relation for a Flat Plate

One of the important aspects of boundary layer theory is the determination of drag caused by the shear force on the body. With respect to flat plate, the drag coefficient is estimated through numerical solution of *Blasius equation* which is a third-order non-linear ordinary differential equation. In order to get rid of the differential equation, there is an alternative method by which the exact prediction of drag coefficient is possible. The *momentum integral method* is one of the alternative techniques by which boundary layer parameters can be predicted through control volume analysis.

Consider a uniform flow past a flat plate and the growth of boundary layer as shown in Fig. 5.10.1. A fixed control volume is chosen for which the uniform flow enters at the leading edge at the section '1'. At the exit of the control volume (section '1'), the velocity varies from zero (at the wall) to the free stream velocity (U) at the edge of the boundary layer. It is assumed that the pressure is constant throughout the flow field. The width and height of the control volume are taken as b and h, respectively.

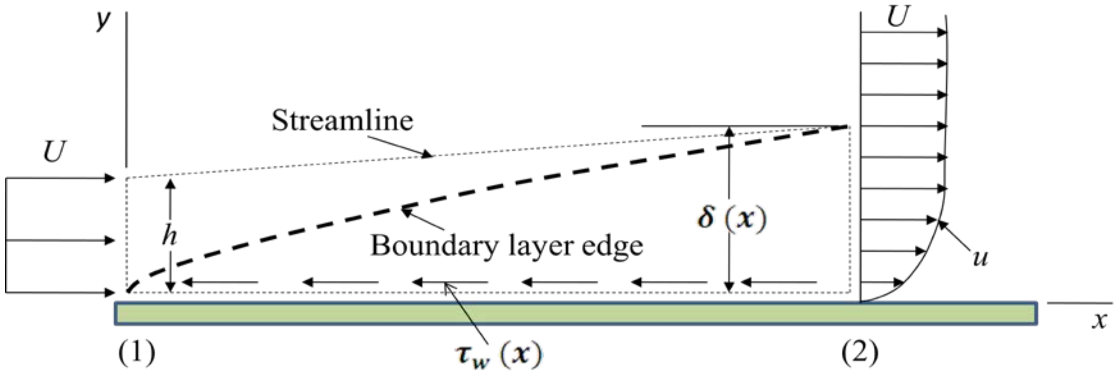

Fig. 5.10.1: Control volume analysis for the flat pate.

The drag force (D) on the plate can be obtained when the x-momentum equation is applied to the steady flow of fluid within this control volume. It is same as the integration of wall shear stress (τ_w) along the length of the plate (Fig. 5.10.1).

$$\sum F_x = -D = -\int \tau_w dA = -l\int \tau_w dx \quad \text{and}$$
$$\sum F_x = \rho\int_1 U(-U)\,dA + \rho\int_2 u^2\,dA \tag{5.10.1}$$

This equation leads to the expression for drag force as given below;

$$D = \rho U^2 bh - \rho l\int_0^\delta u^2 dA \tag{5.10.2}$$

Now, write the mass conservation equation, for the sections '1' and '2'.

$$Uh = \int_0^\delta u\,dA \quad \Rightarrow \rho U^2 bh = \rho b\int_0^\delta U\,u\,dA \tag{5.10.3}$$

The expression for drag force can be obtained by combining Eqs (5.10.2 & 5.10.3).

$$D = \rho b\int_0^\delta u\,(U-u)\,dA \tag{5.10.4}$$

Now, recall the expression for momentum thickness;

$$\theta^* = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy \quad \Rightarrow U^2\theta^* = \int_0^\delta u(U-u)\,dA \tag{5.10.5}$$

So, Eq. (5.10.4) can be rewritten as,

$$D = \rho b U^2\theta^* \quad \Rightarrow \frac{dD}{dx} = \rho b U^2\frac{d\,\theta*}{dx} \tag{5.10.6}$$

It follows from Eqs (5.10.1) that $\dfrac{dD}{dx} = b\,\tau_w$ so that the wall-shear stress can be obtained through Eq. (5.10.6).

$$\tau_w = \rho U^2\frac{d\,\theta*}{dx} \tag{5.10.7}$$

The usefulness of Eq. (5.10.7) lies in the ability to obtain wall shear stress from the velocity layer profile. It is known as the *momentum-integral relation*. Moreover, this equation is valid for laminar as well as turbulent flows.

Solution of Momentum Integral Relation

With the knowledge of a velocity profile within the boundary layer, the momentum integral equation can be used to obtain all the boundary layer parameters. In general, it is appropriate to assume certain velocity from the experimental data. Thus, the accuracy of the results depends on how closely the assumed profile approximates to the actual profile. Let us consider a general velocity profile in a boundary layer;

$$\frac{u}{U} = g(Y) \text{ for } 0 \le Y \le 1 \quad \text{and} \quad \frac{u}{U} = 1 \text{ for } Y > 1 \qquad (5.10.8)$$

Here, the dimensionless parameter $Y = /\delta$ varies from 0 to 1 across the boundary layer and one can assume the dimensionless function of any shape. Let is impose the boundary conditions and write the functions;

$$\begin{aligned} y = 0 &\Rightarrow g(Y) = 0,\ u = 0 \\ y = \delta &\Rightarrow g(Y) = 1,\ u = U,\ (dg/dY) = 0 \end{aligned} \qquad (5.10.9)$$

For a given function $g(Y)$, one can calculate the drag force on the plate from Eq. (5.10.4).

$$\begin{aligned} D &= \rho b \int_0^\delta u(U-u)\,dA = \rho b U^2 \delta \int_0^1 g(Y)\left[1-g(Y)\right] dY = \rho b U^2 \delta C_1 \\ \text{or,}\quad \frac{dD}{dx} &= \rho b U^2 C_1 \left(\frac{d\delta}{dx}\right) = b\tau_w \end{aligned} \qquad (5.10.10)$$

The wall shear stress can also be obtained in the following form;

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\mu U}{\delta}\right)\left(\frac{dg}{dY}\right)_{Y=0} = \left(\frac{\mu U}{\delta}\right) C_2 \qquad (5.10.11)$$

Here, $C_1 = \int_0^1 g(Y)\left[1-g(Y)\right] dY$ is a dimensional constant and is evaluated with assumed velocity profile and $C_2 = \left(\frac{dg}{dY}\right)_{Y=0}$. Combining the Eqs (5.10.10 & 5.10.11) and integrating the resulting expression, one can obtain the following expression;

$$\delta\, d\delta = \left(\frac{\mu}{\rho U}\right)\left(\frac{C_2}{C_1}\right) dx \quad \Rightarrow \frac{\delta}{x} = \frac{\sqrt{2C_2/C_1}}{\sqrt{\mathrm{Re}_x}} \qquad (5.10.12)$$

Substituting Eq. (5.10.12) back into Eq. 5.10.11, the expression of τ_w is obtained;

$$\tau_w = \sqrt{\frac{C_1 C_2}{2}}\, U^{\frac{3}{2}} \sqrt{\frac{\rho\mu}{x}} \qquad (5.10.13)$$

The dimensionless local skin-friction coefficient is obtained as,

$$c_f = \frac{\tau_w}{(1/2)\rho U^2} = \frac{\sqrt{2C_1C_2}}{\sqrt{\mathrm{Re}_x}} \qquad (5.10.14)$$

For a flat plate, of certain length (l) and width (b), the net friction drag is often expressed in terms of friction drag coefficient (C_{Df}).

$$C_{Df} = \frac{D}{(1/2)\rho U^2 bl} = \frac{b\int_0^l \tau_w dx}{(1/2)\rho U^2 bl} = \frac{1}{l}\int_0^l c_f\, dx \quad \Rightarrow C_{Df} = \frac{\sqrt{8C_1C_2}}{\sqrt{\mathrm{Re}_l}} \qquad (5.10.15)$$

It may be observed from the above analysis that the functional dependence of δ and τ_w on the physical parameters is the same for any assumed velocity profile while the constants are different.

$$\frac{\delta}{x}\sqrt{\mathrm{Re}_x} = \sqrt{2C_2/C_1};\ c_f\sqrt{\mathrm{Re}_x} = \sqrt{2C_1C_2};\ C_{Df}\sqrt{\mathrm{Re}_l} = \sqrt{8C_1C_2} \qquad (5.10.16)$$

Several velocity profiles may be assumed for boundary layer as shown in Fig. 5.10.2. The more closely assumed shape with the experimental data for a flat plate is the *Blasius* profile. The non-dimensional constant parameters in Eq. (5.10.16) can be evaluated through the *momentum-integral* results and given in the Table 5.10.1.

Table 5.10.1: Momentum integral estimates for a laminar flow velocity profiles

Nature of velocity profile	*Equation*	$\frac{\delta}{x}\sqrt{\mathrm{Re}_x}$	$c_f\sqrt{\mathrm{Re}_x}$	$C_{Df}\sqrt{\mathrm{Re}_l}$
Blasius	$\frac{u}{U} = f'\left(\frac{y}{\delta}\right)$	5	0.664	1.328
Linear	$\frac{u}{U} = \frac{y}{\delta}$	3.46	0.578	1.156
Parabolic	$\frac{u}{U} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2$	5.48	0.73	1.46
Cubic	$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	4.64	0.646	1.292
Sine wave	$\frac{u}{U} = \sin\left[\frac{\pi}{2}\left(\frac{y}{\delta}\right)\right]$	4.79	0.655	1.31

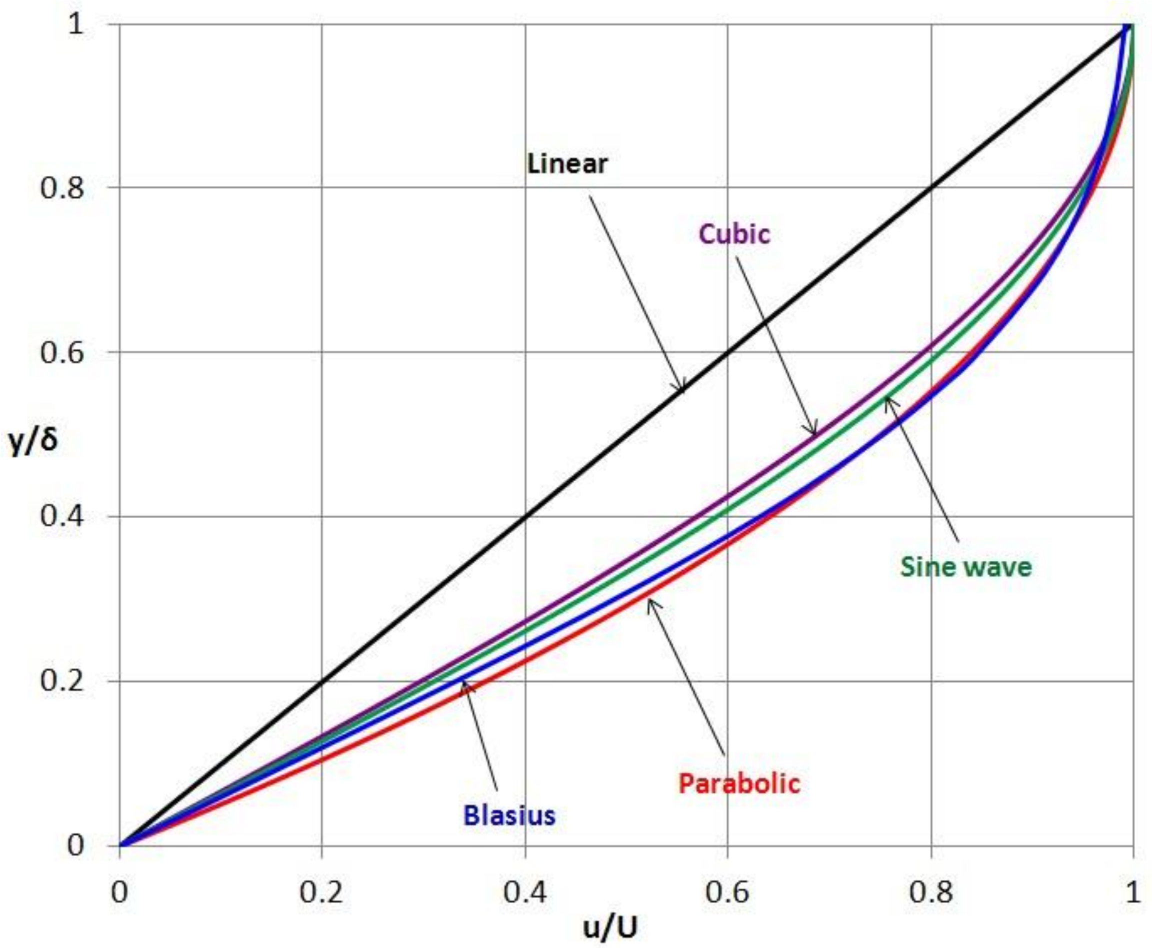

Fig. 5.10.2: Approximate boundary layer profile for momentum integral estimates.

Module 5 : Lecture 11
VISCOUS INCOMPRESSIBLE FLOW
(External Flow – Part V)

Turbulent Flat Plate Boundary Layer

A laminar boundary layer over a flat plate eventually becomes turbulent over certain range of Reynolds number. There is no unique value pf Reynolds number, for this change to happen. It mainly depends on the free stream turbulence and surface roughness parameters. With a very fine polished wall and with a quiet free stream, one can delay the transition. A controlling parameter such as the critical *Reynolds number of transition* $\left(\mathrm{Re}_{x,CR}\right)$ may be defined. On a flat plate with a sharp leading edge in a typical free stream air flow, the transition occurs between the Reynolds number ranges of 2×10^5 to 3×10^6. So the transitional Reynolds number is normally taken as $\mathrm{Re}_{x,CR} = 5\times10^5$.

The complex process of transition from laminar to turbulent flow involves the instability in the flow field. The small disturbances imposed on the boundary layer flow will either grow (i.e. instability) or decay (stability) depending on the location where the disturbance is introduced. If the disturbance occurs at a location where $\mathrm{Re}_x < \mathrm{Re}_{x,CR}$, then the boundary layer will return to laminar flow at that location. Disturbances imposed on locations $\mathrm{Re}_x > \mathrm{Re}_{x,CR}$ will grow and the boundary layer flow becomes turbulent from this location. The transition to turbulence involves noticeable change in the shape of boundary layer velocity profile as shown in Fig. 5.11.1. As compared to laminar profiles, the turbulent velocity profiles are flatter and thicker at the same Reynolds number (Fig. 5.11.2). Also, they have larger velocity gradient at the wall.

There is no exact theory for turbulent flat plate flow rather many empirical models are available. To begin with the analysis of turbulent boundary layer, let us recall the momentum-integral relation which is valid for both laminar as well as turbulent flows.

$$\tau_w(x) = \rho U^2 \frac{d\theta_*}{dx} \qquad (5.11.1)$$

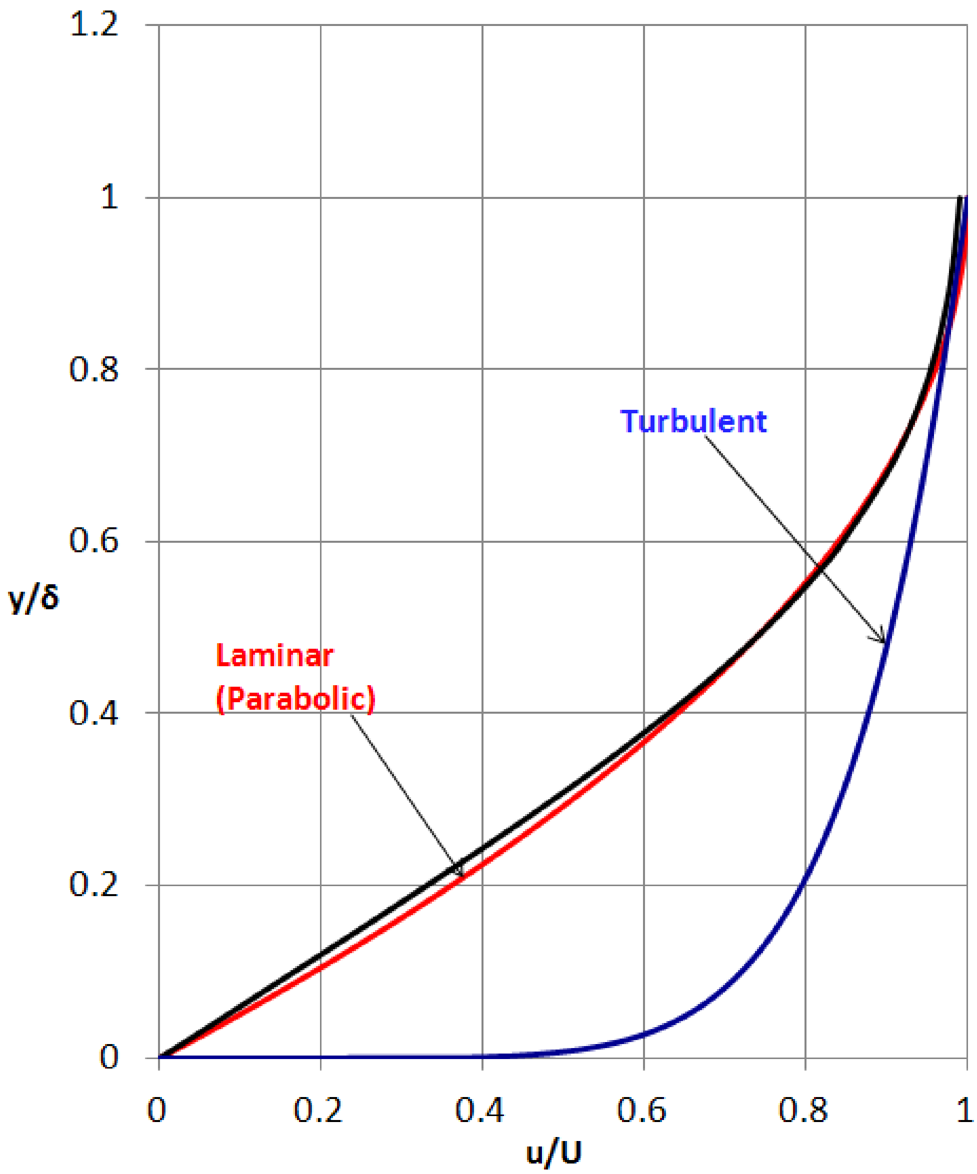

Fig. 5.11.1: Comparison of laminar and turbulent boundary layer profiles for flat plate.

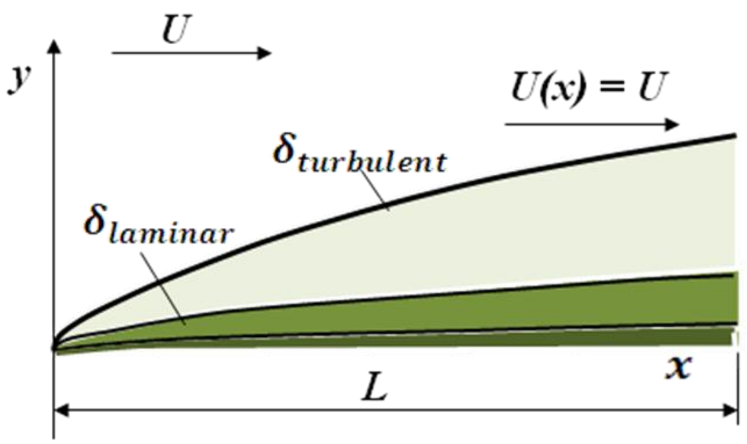

Fig. 5.11.2: Comparison of laminar and turbulent boundary layer profiles for flat plate.

In order to obtain the momentum thickness in Eq. (5.11.1), it is desired to know the velocity profile. *Prandtl* suggested the *power law* approximation for skin friction coefficient and *one-seventh power law* for velocity profile which is considered as very good approximations for flat plates.

$$c_f \approx 0.02\left(\mathrm{Re}_\delta\right)^{-(1/6)}$$
$$\frac{u}{U} \approx \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \text{ for } y \le \delta \text{ and } \frac{u}{U} \approx 1 \text{ for } y > \delta \qquad (5.11.2)$$

The approximate turbulent velocity profile shape given by Eq. (5.11.2) leads to the fact that the slope $\left(\partial u / \partial y\right)_{y=0}$ is infinite at the wall which is not meaningful physically.

However, the large slope leads to a very high skin friction on the surface of the plate as compared to the laminar flow under similar conditions. With this approximate profile, the momentum thickness can be easily evaluated:

$$\theta^* \approx \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right] dy = \frac{7}{72}\delta \qquad (5.11.3)$$

From the definition of skin friction coefficient and using the Eq. (5.11.1), the results are rewritten as below;

$$c_f = \frac{\tau_w(x)}{(1/2)\rho U^2} = 2\frac{d\theta^*}{dx} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right) \qquad (5.11.4)$$

Substituting Eq. (5.11.4) in Eq. (5.11.2), separating the variables and integrating the resulting expression by assuming $\delta = 0$ at $x = 0$, the following important relation is obtained for boundary layer thickness.

$$\left(\mathrm{Re}_\delta\right) = 0.16\left(\mathrm{Re}_x\right)^{6/7} \Rightarrow \frac{\delta}{x} = \frac{0.16}{\left(\mathrm{Re}_x\right)^{1/7}} \qquad (5.11.5)$$

It may be observed that the thickness of a turbulent boundary layer increases as $x^{6/7}$ while it increases as $x^{1/2}$ for a laminar boundary layer. It means that a turbulent boundary layer grows at a faster rate compared to that of a laminar boundary layer. Eq. (5.11.5) is the solution of a turbulent boundary layer because all the boundary layer parameters can be obtained from this equation as given below;

- Displacement thickness,

$$\delta^* = \int_0^\delta \left(1-\frac{u}{U}\right)dy = \int_0^\delta \left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy = \frac{\delta}{8} = \left(\frac{1}{8}\right)\frac{0.16x}{(\mathrm{Re}_x)^{1/7}} \Rightarrow \frac{\delta_*}{x} = \frac{0.02}{(\mathrm{Re}_x)^{1/7}} \quad (5.11.6)$$

- Momentum thickness,

$$\theta^* = \frac{7}{72}\delta = \frac{7}{72}\left[\frac{0.16\,x}{(\mathrm{Re}_x)^{1/7}}\right] \Rightarrow \frac{\theta^*}{x} = \frac{0.016}{(\mathrm{Re}_x)^{1/7}} \quad (5.11.7)$$

- Turbulent shape factor for flat plate

$$H = \frac{\delta_*}{\theta^*} = 1.25 \quad (5.11.8)$$

- Skin friction coefficient,

$$c_f = 0.02(\mathrm{Re}_\delta)^{-(1/6)} = 0.02\left[0.16\,(\mathrm{Re}_x)^{6/7}\right]^{-(1/6)} = \frac{0.027}{(\mathrm{Re}_x)^{1/7}} \quad (5.11.9)$$

- Wall shear stress,

$$c_f = \frac{\tau_w}{(1/2)\rho U^2} = \frac{0.027}{(\mathrm{Re}_x)^{1/7}} \Rightarrow \tau_w = \frac{0.0135\mu^{1/7}\rho^{6/7}U^{13/7}}{x^{1/7}} \quad (5.11.10)$$

- Drag coefficient,

$$c_d = \frac{1}{L}\int_0^L c_f\,dx = \frac{0.031}{(\mathrm{Re}_L)^{1/7}} \quad (5.11.11)$$

These are some basic results of turbulent flat plate theory. The flat plate analysis for a *Blasius* laminar boundary layer and turbulent boundary layer is summarized in Table 5.11.1.

Table 5.11.1: Comparative analysis of laminar and turbulent boundary layer flow over a flat plate

Parameters	*Laminar (Blasius solution)*	*Turbulent (Prandtl approximation)*
Boundary layer thickness	$\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} = \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} = \frac{0.02}{(\mathrm{Re}_x)^{1/7}}$
Momentum thickness	$\frac{\theta^*}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta^*}{x} = \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$
Shape factor	$H = \frac{\delta^*}{\theta^*} = 2.59$	$H = \frac{\delta^*}{\theta^*} = 1.25$
Local skin friction coefficient	$c_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$c_f = \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$
Wall shear stress	$\tau_w = \frac{0.332\mu^{1/2}\rho^{1/2}U^{3/2}}{x^{1/2}}$	$\tau_w = \frac{0.0135\mu^{1/7}\rho^{6/7}U^{13/7}}{x^{1/7}}$
Drag coefficient	$c_d = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$	$c_d = \frac{0.031}{(\mathrm{Re}_L)^{1/7}}$

Effect of Pressure Gradient on the Boundary Layer

The analysis of viscous flow fields past an external body (such as flat plate) is essentially done by dividing the entire flow domain in two parts; outer inviscid flow and a boundary layer flow which is predominant in the thin region close to the surface of the plate. Depending on the nature of boundary layer (laminar/turbulent), the velocity profile and all other relevant parameters are determined. However, when the outer flow accelerates/decelerates, few interesting phenomena take place within the boundary layer. If the outer inviscid and/or irrotational flow accelerates, $U(x)$ increases and using Euler's equation, it may be shown that $p(x)$ decreases. The boundary layer in such an accelerating flow is formed very close to the wall, usually thin and is not likely to separate. Such a situation is called as *favorable pressure gradient* $\left(\frac{dp}{dx} < 0\right)$. In the reverse case, when the outer flow decelerates, $U(x)$ decreases and $p(x)$ increases leading to *unfavorable/adverse pressure gradient* $\left(\frac{dp}{dx} > 0\right)$. This condition is not desirable because the boundary layer is usually thicker and does not stick to the wall. So, the flow is more likely to separate from the wall due to excessive momentum loss to counteract the effects of adverse pressures. The *separation* leads to the *flow reversal* near the wall and destroys the parabolic nature of the flow field. The boundary layer equations are not valid downstream of a separation point because of the reverse flow in the separation region. Let us explain the phenomena of separation in the mathematical point of view. First recall the boundary layer equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} = U\frac{dU}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{5.11.12}$$

When the separation occurs, the flow is no longer attached to the wall i.e. $u = v = 0$. Then, Eq. (5.11.12) is simplified and is valid for either laminar/turbulent flows.

$$\begin{aligned} &\left(\frac{\partial \tau}{\partial y}\right)_{wall} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = -\rho U\frac{dU}{dx} = \frac{dp}{dx} \\ &\text{or, } \left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = \frac{1}{\mu}\left(\frac{dp}{dx}\right) \end{aligned} \tag{5.11.13}$$

From the nature of differential equation (Eq. 5.11.13), it is seen that the second derivative of velocity is positive at the wall in the case of adverse pressure gradient. At the same time, it must be negative at the outer layer $(y = \delta)$ to merge smoothly with the main stream flow $U(x)$. It follows that the second derivative must pass through *zero* which is known as the *point of inflection* (PI) and any boundary layer profile in an adverse gradient situation must exhibit a characteristic *S-shape*. The effect of pressure gradient on the flat plate boundary layer profile is illustrated below and is shown in Fig. 5.11.3.

Case I: Under the *favorable pressure gradient* conditions $\left(\frac{dp}{dx} < 0;\ \frac{dU}{dx} > 0; \frac{\partial^2 u}{\partial y^2} < 0\right)$, the velocity profile across the boundary layer is rounded without any *inflection point* (Fig. 5.11.3-a). No separation occurs in this case and u approaches to $U(x)$ at the edge of the boundary layer. The wall shear stress (τ_w) is the largest compared to all other cases

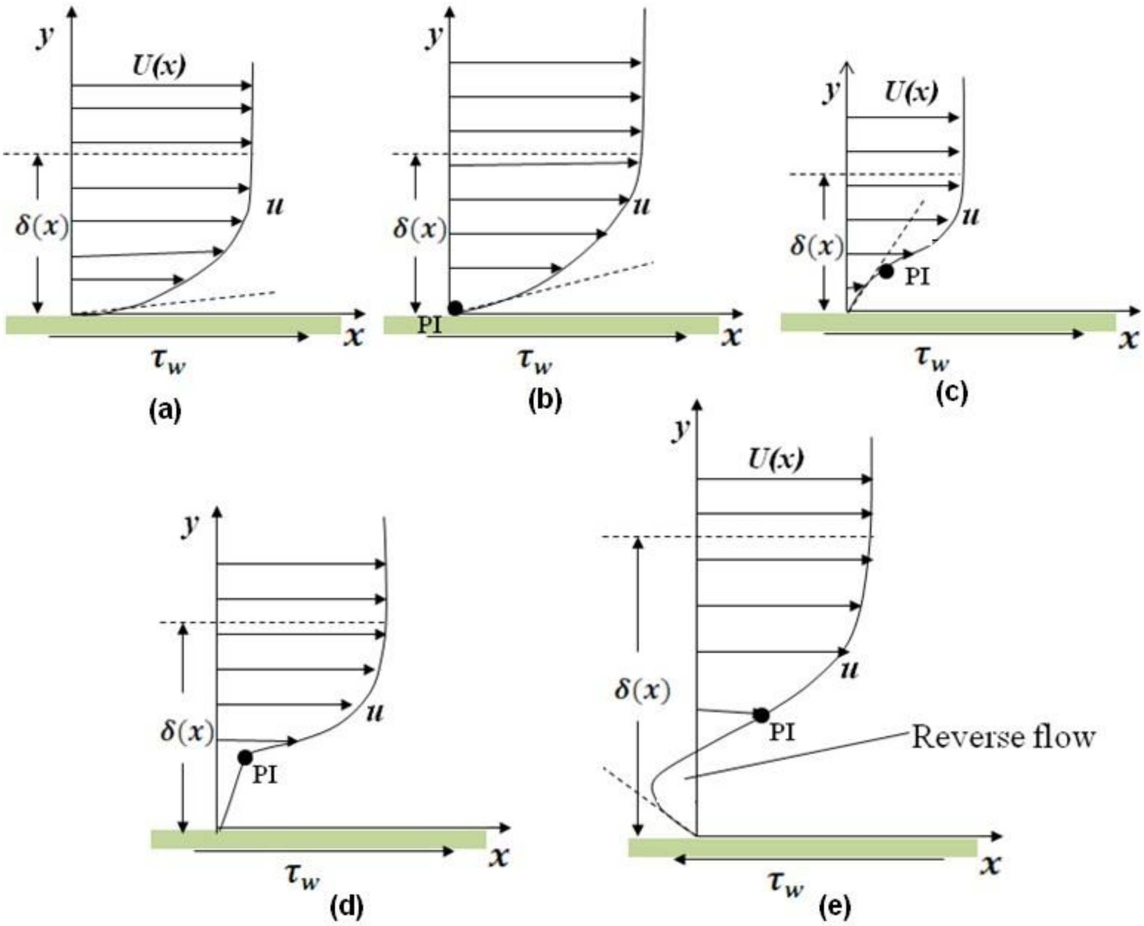

Fig. 5.11.3: Effect of pressure gradient on the boundary layer for a flat plate.

Case II: When *pressure gradient* is zero, $\left(\frac{dp}{dx}=0;\ \frac{dU}{dx}=0;\frac{\partial^2 u}{\partial y^2}=0\right)$, the *point of inflection* lies on the wall itself and there is no separation (Fig. 5.11.3-b). It implies a linear growth of u with respect to y for the boundary layer profile and is same as the *Blasius* boundary layer profile for the flat plate. The flow has a tendency to undergo the transition in the Reynolds number of about 3×10^6.

Case III: In a situation of *adverse pressure gradient* , $\left(\frac{dp}{dx}>0;\ \frac{dU}{dx}<0;\frac{\partial^2 u}{\partial y^2}>0\right)$, the outer flow is decelerated. However, the value of $\left(\partial^2 u/\partial y^2\right)$ must be negative when u approaches to $U(x)$ at the edge of the boundary layer. So, there has to be a point of inflection $\left(\frac{\partial^2 u}{\partial y^2}=0\right)$ somewhere in the boundary layer and the profile looks similar to *S-shape*. In a weak adverse pressure gradient (Fig. 5.11.3-c), the flow does not actually separate but vulnerable to transition to turbulence even at lower Reynolds number of 10^5. At some moderate adverse pressure gradient, the wall shear stress is exactly zero $\left(\tau_w=\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}=0\right)$. This is defined as *separation point* as shown in Fig. 5.11.3(d). Any stronger pressure gradient will cause back flow at the wall that leads to thickening the boundary layer, breaking the main flow and flow reversal at the wall (Fig. 5.11.3-e). Beyond the separation point, the wall shear stress becomes negative $(\tau_w<0)$ and the boundary layer equations break down in the region of separated flow.