# UNIT III – APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially displaced in the position $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$ and then released from rest. Find the displacement $y$ at any distance $x$ from one end at time $t$.

Sol:


The displacement $y(x,t)$ is from $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The Conditions are

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

(iv) $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

$$= \frac{y_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

The Suitable Sol. is

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat) \quad \text{--- (1)}$$

Apply (i) $y(0,t) = 0$

$$C_1 (C_3 \cos pat + C_4 \sin pat) = 0$$

$$\therefore \boxed{C_1 = 0}$$

Apply (ii) in (1) $x = l$

$$y(l,t) = C_2 \sin pl\,(C_3 \cos pat + C_4 \sin pat)$$

$$C_2 \sin pl = 0$$

$$\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = n\pi/l.$$

(1) Becomes

$$y(x,t) = B_2 \sin\frac{n\pi x}{l}\left(C_3 \cos\frac{n\pi a t}{l} + C_4 \sin\frac{n\pi a t}{l}\right) \quad \text{--- (2)}$$

~~The most general soln. is~~

~~$y(x,t) = \sum_{n=1}^{\infty}$~~

now apply cond. (iii) we get

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = C_2 \sin\left(\frac{n\pi x}{l}\right) C_4 \frac{n\pi a}{l} = 0$$

$$C_4 = 0$$

$$y(x,t) = C_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

The most general soln.

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi at}{l}\right)$$

Apply (iv) $y(x,0) = f(x)$

$$\frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{2\pi x}{l}\right) + C_3 \sin\left(\frac{3\pi x}{l}\right) + \dots$$

On Comparing

$$\frac{3y_0}{4} = C_1 \quad C_2 = 0 \quad C_3 = -\frac{y_0}{4}$$

$$C_4 = C_5 = \dots = 0.$$

$$y(x,t) = \frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi at}{l}\right) - \frac{y_0}{4} \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi at}{l}\right)$$

A String of length $2l$ is fastened at both ends. The mid pt. of the String is taken to a height b and then released from rest in that Position. Show that the displacement is

— x —

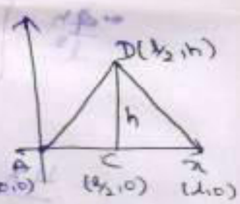
A tightly stretched String of length $l$ has its ends fastened at $x=0$ and $x=l$. The mid point of the String is then taken to a height h and then released from rest in that position. Obtain an expression for the displacement of the String at any Subsequent time.

Sol: The eq. to be Solved is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the gn. problem, We get the following boundary and initial Conditions.

(1) $y(0,t)=0$ (2) $y(l,t)=0$ (3) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)}=0$

Eq. of AD is



$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \Rightarrow \frac{x-0}{l/2-0} = \frac{y-0}{h-0}$$

$$xh = \frac{l}{2} y \Rightarrow y = \frac{2}{l} xh$$

Eq. of DB is

$$\frac{x-l/2}{l-l/2} = \frac{y-h}{0-h} \Rightarrow \frac{x-l/2}{l/2} = \frac{y-h}{-h}$$

$$y-h = -\frac{2h}{l}(x-l/2) \Rightarrow y = h - \frac{2hx}{l} + h$$

$$y = 2h(1-x/l) = \frac{2h}{l}(l-x)$$

(4) $y(x,0) = \begin{cases} \frac{2hx}{l} & 0 < x < l/2 \\ \frac{2h}{l}(l-x) & l/2 < x < l \end{cases}$

(5) $y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$

Applying cond. (ii) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = \begin{cases} \frac{2hx}{l} & 0<x<l/2 \\ \frac{2h(l-x)}{l} & l/2<x<l. \end{cases}$$

To find $C_n$ :- Expand the value in a half-range Sine Series

$$\left.\begin{array}{l} \frac{2hx}{l}, \quad 0<x<l/2 \\ \frac{2h}{l}(l-x), \quad l/2<x<l \end{array}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$C_n = \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4h}{l^2}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4h}{l^2}\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l}\right]_0^{l/2}$$

$$+ \frac{4h}{l^2}\left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - (l/n\pi)^2 \sin\frac{n\pi x}{l}\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left[\left(\frac{-l^2}{2n\pi}\cos n\pi/2 + (l/n\pi)^2 \sin n\pi/2\right) - (0+0)\right]$$

$$+ \frac{4h}{l^2}\left[(-0-0) - \left(\frac{-l^2}{2n\pi}\cos n\pi/2 - (l/n\pi)^2 \sin n\pi/2\right)\right]$$

$$= \frac{4h}{l^2}\left[\frac{-l^2}{2n\pi}\cos(n\pi/2) + (l/n\pi)^2 \sin n\pi/2 + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + (l/n\pi)^2 \sin n\pi/2\right]$$

$$= \frac{8h}{n^2\pi^2} \sin\frac{n\pi}{2}$$

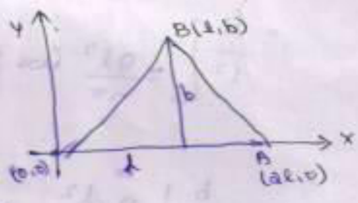
$$y(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$$

— x —

$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \cos\left(\frac{(2n-1)\pi at}{2l}\right)$$

Sol: The Wave eq/ is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

1. $y(0,t) = 0$
2. $y(2l,t) = 0$
3. $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0$




Eq/ of OB

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{b-0} = \frac{x-0}{l-0} \quad y = \frac{b}{l}x.$$

Eq of BA is

$$\frac{y-b}{0-b} = \frac{x-l}{2l-l} \Rightarrow y = b\left(\frac{2l-x}{l}\right)$$

$$y(x,0) = \begin{cases} \frac{bx}{l} \\ \frac{b(2l-x)}{l} \end{cases}$$

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat)$$

$$y(0,t) = 0$$
$$\Rightarrow c_1 = 0$$

$$\boxed{p = n\pi/2l}$$

$$c_4 = 0$$

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{2l} = \begin{cases} \frac{bx}{l} \\ \frac{b(2l-x)}{l} \end{cases}$$

To find $C_n$ expand the Gn/ value in a half-range Fourier Sine Series in the interval $(0,L)$ $(L = 2l)$

$$b_n = \frac{2}{(2l)} \int_0^{2l} f(x) \sin\frac{n\pi x}{2l}\, dx.$$

$$\therefore C_n = \frac{b}{l^2}\left[\int_0^l x \sin\frac{n\pi x}{2l}\,dx + \int_l^{2l} (2l-x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{b}{l^2}\left[-x\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} + \left(\frac{2l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2l}\right)\right]_0^l$$

$$+\frac{b}{l^2}\left[-(2l-x)\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} - \left(\frac{2l}{n\pi}\right)^2 \sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left(\frac{8l^2}{n^2\pi^2}\sin n\pi/2\right)$$

$$C_n = \frac{8b}{n^2\pi^2}\sin n\pi/2$$

$= 0$ if $n$ is even

$= \dfrac{8b}{n^2\pi^2}\sin n\pi/2$ $n$ is odd

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8b}{n^2\pi^2}\sin n\pi/2 \,\sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}$$

$$= \sum_{n=1}^{\infty}\frac{8b}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi a t}{2l}$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}(-1)^{n-1}\sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi a t}{2l}$$

Problems on Vibrating String with non-zero initial velocity.

The boundary and initial Conditions of the deflection $y(x,t)$ are

(1) $y(0,t)=0$ (2) $y(l,t)=0$ (3) $y(x,0)=0$ (4) $\frac{\partial y}{\partial t}(x,0)=f(x)$

The suitable soln. is

$y(x,t)=(C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat)$ — (1)

Apply cond. (1) we get $C_1=0$

(2) $p=n\pi/l$

(3) $C_3=0$

The most general soln. is

$$y(x,t)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

Apply (4) Con.

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)}=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}=f(x).$$

where $B_n = C_n \frac{n\pi a}{l}$

$$B_n=b_n=\frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx.$$

$$\therefore C_n=\frac{l}{n\pi a} B_n.$$

A tightly stretched string with fixed end pts. $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating string giving each point a velocity $\lambda x(l-x)$ Show that the displacement is

$$y(x,t)=\frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}$$

Soln: The wave eqn is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Boundary & initial Condn.

(1) $y(0,t)=0 \ \forall t$

(2) $y(l,t)=0$ (3) $y(x,0)=0 \quad 0<x<l$

(4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \lambda x(l-x) \quad 0<x<l$.

The Suitable soln is

$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat)$ → (I)

Apply Condn (1) in I

$y(0,t) = C_1 [C_3 \cos pat + C_4 \sin pat) = 0$

$C_3 \cos pat + C_4 \sin pat \neq 0$

$\therefore \boxed{C_1 = 0}$

Substitute $C_1=0$ in (I)

$y(x,t) = C_2 \sin px (C_3 \cos pat + C_4 \sin pat)$ → (II)

Apply Condn (2) in II

$y(l,t) = C_2 \sin pl (C_3 \cos pat + C_4 \sin pat) = 0$

Here $C_3 \cos pat + C_4 \sin pat \neq 0$.

$\cancel{C_2 \sin pl}$: either $C_2=0$ (or) $\sin pl = 0$

Suppose $C_2=0$ already we have $C_1=0$.

$\therefore C_2 \neq 0$. $\sin pl = 0 = \sin n\pi$

$pl = n\pi \Rightarrow p = n\pi/l$.

Now, Sub $p = n\pi/l$ in eqn II

$y(x,t) = C_2 \sin \frac{n\pi x}{l}\left(C_3 \cos \frac{n\pi at}{l} + C_4 \sin \frac{n\pi at}{l}\right)$ → (III)

Apply Condn (3) in III

$y(x,0) = C_2 \sin \frac{n\pi x}{l} C_3 = 0$.

$C_2 C_3 \sin \frac{n\pi x}{l} = 0$

$\sin \frac{n\pi x}{\lambda} \neq 0$ (it is defined for all n)

$C_2 \neq 0$.

$\therefore C_3 = 0$.

Sub $C_3 = 0$ in eq B

$y(x,t) = C_2 C_4 \sin \frac{n\pi x}{\lambda} \sin \frac{n\pi a t}{l}$

$= C_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{\lambda}$ — IV

The most general soln.

$y(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\lambda} \sin \frac{n\pi a t}{l}$ — X

Before apply condn. diff X

$\left(\frac{\partial y}{\partial t}\right)_{(x,t)} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\lambda} \left(\frac{n\pi a}{\lambda}\right) \cos \frac{n\pi a t}{\lambda}$

apply (4) condn.

$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{\lambda} \left(\frac{n\pi a}{\lambda}\right) = \lambda x(l-x)$

$= \sum_{n=1}^{\infty} \left(C_n \frac{n\pi a}{\lambda}\right) \sin \frac{n\pi x}{\lambda} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\lambda}$

$B_n = C_n \frac{n\pi a}{l}$

To find $B_n$:- Expand $\lambda x(l-x)$ in a half range. Sine Series

$\lambda x(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\lambda}$

$b_n = \frac{2}{\lambda} \int_0^{\lambda} f(x) \sin \frac{n\pi x}{\lambda} dx$.

$B_n = \frac{2}{l} \int_0^{l} \lambda x(l-x) \sin \frac{n\pi x}{\lambda} dx$.

$$= \frac{2\lambda}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l} + (l-2x)\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi x}{l} - 2\left(\frac{l}{n\pi}\right)^3\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\left(-2\left(\frac{l}{n\pi}\right)^3(-1)^n\right) - \left(-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$= \frac{2\lambda}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n + 2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2\lambda}{l} \times 2\left(\frac{l}{n\pi}\right)^3\left[1-(-1)^n\right]$$

$$= \frac{4\lambda l^2}{n^3\pi^3}\left(1-(-1)^n\right) = C_n\left(\frac{n\pi a}{l}\right)$$

$$C_n = \left(\frac{l}{n\pi a}\right)\left(\frac{4\lambda l^2}{n^3\pi^3}\right)\left(1-(-1)^n\right)$$

$$C_n = \begin{cases} 0 & n = \text{even} \\ \dfrac{8\lambda l^3}{a n^4\pi^4} & n = \text{odd} \end{cases}$$

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=\text{odd}}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}.$$

— x —

(2) A tightly stretched string of length $l$ is initially at rest in its equilibrium position and each of its pts is given the velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right)$. Find the displacement.

Sol: The wave eqn. is $a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

The condn. are (1) $y(0,t) = 0$ (2) $y(l,t) = 0$

(3) $y(x,0) = 0$ (4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = V_0 \sin^3\left(\frac{\pi x}{l}\right)$

$$\frac{3V_0}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{V_0}{4}\sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$= B_1\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right) + B_2\sin\left(\frac{2\pi x}{l}\right)\left(\frac{2\pi a}{l}\right) + B_3\left(\frac{3\pi a}{l}\right)\sin\left(\frac{3\pi x}{l}\right) + \cdots$$

$B_1 = \frac{3V_0}{4}$ $\quad B_3 = -\frac{V_0}{4}$.

$B_1 = C_1\left(\frac{\pi a}{l}\right) \Rightarrow \frac{3V_0}{4}\left(\frac{l}{\pi a}\right) = C_1$

$\frac{3V_0 l}{4\pi c} = C_1$

$B_3 = C_3\left(\frac{3\pi a}{l}\right) = -\frac{V_0}{4}$

$C_3 = -\frac{lV_0}{12\pi a}$

$$y(x,t) = \frac{3V_0 l}{4\pi c}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi a t}{l}\right) - \frac{V_0 l}{12\pi a}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi a t}{l}\right)$$

(8) A string of length $l$ is initially at rest in its equilibrium position and motion its started by giving each of its pts a velocity gn. by

$$V = \begin{cases} cx & \text{if } 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l. \end{cases}$$ Find the displacement

Soln: The wave eqn.

boundary Condn.

(1) $y(0,t) = 0$ (2) $y(l,t) = 0$ (3) $y(x,0) = 0$

(4) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l. \end{cases}$

$y(x,t) = (C_1\cos px + C_2\sin px)(C_3\cos pat + C_4\sin pat)$

Apply Cond (4)

$$\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a}{l}\right) = \begin{cases} cx & 0 \le x \le l/2 \\ c(l-x) & l/2 \le x \le l \end{cases}$$

$$B_n = C_n\left(\frac{n\pi a}{l}\right)$$

To find $B_n$.

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \qquad (6) \text{ \& } (7) \quad b_n = B_n$$

$$= \frac{2c}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l} dx + \int_{l/2}^{l} (l-x) \sin\frac{n\pi x}{l} dx\right]$$

$$= \frac{2c}{l}\left[\left[-x\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right)\right]_0^{l/2} + \left[-(l-x)\left(\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right)\right]_{l/2}^{l}\right]$$

$$= \frac{2c}{l}\left[-\frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{l^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2c}{l}\left(\frac{2l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi}{2}\right) = \frac{4lc}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$C_n = \frac{l}{n\pi a} B_n = \frac{l}{n\pi a}\left(\frac{4lc}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4l^2c}{n^3\pi^3 a}\sin\left(\frac{n\pi}{2}\right)\cdot\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi at}{l}\right)$$

One dimensional Heat eqn.

The One dimensional heat eqn is

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad -①$$

$\alpha^2 = \frac{K}{\rho c} \rightarrow$ Diffusivity of the material

$k \rightarrow$ Thermal Conductivity

$\rho \rightarrow$ Density

$c \rightarrow$ Specific heat.

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Temperature fn. is $U(x,t)$

CIVIL

— x —

Note: $\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$A = \alpha^2 \quad B = 0 \quad C = 0$

$B^2 - 4AC = 0 = 0 - 4\alpha^2(0) = 0$

The One dimensional heat eqn is parabolic

Write all possible soln. of one dimensional heat eqn.

① $u(x,t) = [Ae^{px} + Be^{-px}]\, C e^{\alpha^2 p^2 t}$

② $u(x,t) = [A\cos px + B\sin px]\, C e^{-\alpha^2 p^2 t}$

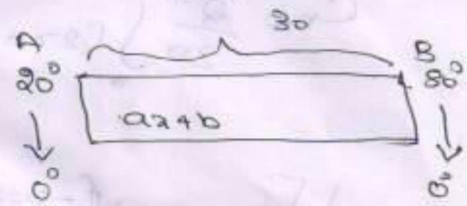
③ $u(x,t) = (Ax + B)C.$

— ^ —

A rod 30 cm long has its ends A and B kept at 20°c and 80°c respectively. Until steady state conditions prevail, the temp at each end is then suddenly reduced to 0°c and kept so on. Find the resulting

Sol: The temp. $u(x,t)$ is from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$




The Cond. are

(1) $u(0,t)=0$

(2) $u(30,t)=0$

(3) $u(x,0)=ax+b = 2x+20$

$a = \frac{80-20}{30} = \frac{60}{30} = 2$

$b = \cancel{\frac{0+0}{30}} = 0$

$b = 20$

The Soln.

$u(x,t) = [A\cos px + B\sin px]\, C e^{-\alpha^2 p^2 t}$ — (1)

Apply (1) put $x=0$

$0 = A C e^{-\alpha^2 p^2 t}$

$\boxed{A=0}$

Apply (ii) Put $x=30$

$0 = B\sin(30p)\, C e^{-\alpha^2 p^2 t}$

$B\sin(30p)=0$

$\sin(30p)=0 \Rightarrow 30p = n\pi \Rightarrow p = n\pi/30$

$\therefore$ (1) Becomes

$$u(x,t) = B.\sin\left(\frac{n\pi x}{30}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{90}}$$

The most general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{900}}$$

Apply (iii) $[$Put $t=0]$

$$2x+20 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{30}\right)$$

$$C_n = \frac{2}{30}\int_0^{30} (2x+20)\sin\left(\frac{n\pi x}{30}\right)dx$$

$$= \frac{1}{15}\left[(2x+20)\left\{-\cos\left(\frac{n\pi x}{30}\right)\left(\frac{30}{n\pi}\right)\right\} - 2\left\{-\sin\left(\frac{n\pi x}{30}\right)\left(\frac{900}{n^2\pi^2}\right)\right\}\right]_0^{30}$$

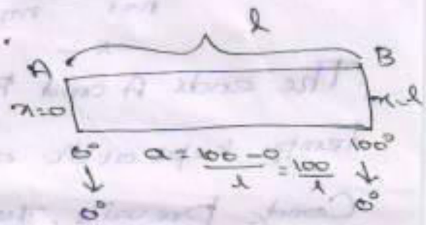
$$= \frac{1}{15}\left[-\frac{2400}{n\pi}(-1)^n + \frac{600}{n\pi}(1)\right]$$

$$= \frac{600}{15n\pi}\left[-4(-1)^n + 1\right]$$

$$= \frac{40}{n\pi}\left[-4(-1)^n + 1\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[-4(-1)^n + 1\right]\sin\left(\frac{n\pi x}{30}\right) e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}.$$

A rod of length $l$ has its ends A and B kept at $0°c$ and $100°c$ respr. Until steady state conditions prevail. If the tempr. at B is reduced suddenly to $0°c$ and kept so while that of A is maintained find the temperature $u(x,t)$.




Sol:

The tempr. $u(x,t)$ is from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

The condn. are

(1) $u(0,t)=0$ (2) $u(l,t)=0$ (3) $u(x,0) = ax+b = \frac{100}{l}x$.

The suitable soln. is

$$u(x,t) = (A\cos px + B\sin px)\, C e^{-\alpha^2 p^2 t}$$

Apply (1) $x=0$

$$A(Ce^{-\alpha^2p^2t}) = 0 \Rightarrow A = 0$$

Apply (2)

$$u(l,t) = \cancel{A\cos pl}(B\sin pl)(Ce^{-\alpha^2p^2t}) = 0$$

$$\sin pl = 0 = \sin n\pi \Rightarrow pl = n\pi \quad \boxed{p = n\pi/l}$$

$$u(x,t) = B\sin\frac{n\pi x}{l}\; C e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

The most genr. $u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$ —(4)

Apply (3) in (4)

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$C_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right)dx.$$

$$= \frac{200}{l^2}\int_0^l x \sin\left(\frac{n\pi x}{l}\right)dx.$$

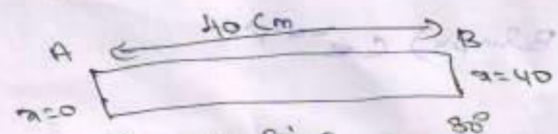
$$= \frac{200}{l^2}\left[x\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right] = \frac{-200(-1)^n}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-200(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

—x—

The ends A and B of a rod 40cm long have their temp. kept at $0^\circ C$ and $80^\circ C$ resp. until steady state cond. prevails, the temp. of the end B is then suddenly reduced to $40^\circ C$ and kept so. while that of the end A is kept at $0^\circ C$ find the subsequent temp. distribution $u(x,t)$ in the rod.




$0^\circ$ $\quad a = \frac{80-0}{40} = 2$

$\downarrow$ $\quad b = 0$

$0^\circ$ $\quad ax+b = 2x$ $\quad 40^\circ$

$$\frac{40-0}{40} = 1$$

$$ax+b = x.$$

The temp. $u(x,t)$ is from $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

The cond. are,

(1) $u(0,t)=0$ (2) $u(40,t)=40$ (3) $u(x,0)=ax+b=2x$.

The suitable sol.

$$u(x,t) = (A\cos px + B\sin px)\, C e^{-\alpha^2 p^2 t} + u_s(x)$$

$$u_s(x) = ax+b = x.$$

$$u(x,t) = x + (A\cos px + B\sin px)\, C e^{-\alpha^2 p^2 t} \quad \text{—(1)}$$

Apply (1)

$$A(C e^{-\alpha^2 p^2 t}) \Rightarrow A = 0$$

Apply ②

$$40 = 40 + (B \sin 40p)\, C e^{-\alpha^2 p^2 t}$$

$$(B \sin 40p)\, C e^{-\alpha^2 p^2 t} = 0$$

$$\sin 40p = \sin n\pi \qquad p = n\pi/40.$$

$$u(x,t) = x + B \sin\left(\frac{n\pi x}{40}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Most genl. soln. is

$$u(x,t) = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Apply (3) put $t=0$

$$2x = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$x = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$C_n = \frac{2}{40}\int_0^{40} x \sin\left(\frac{n\pi x}{40}\right) dx.$$

$$= \frac{1}{20}\left[x\left(-\cos\left(\frac{n\pi x}{40}\right)\left(\frac{40}{n\pi}\right)\right) + \sin\left(\frac{n\pi x}{40}\right)\left(\frac{1600}{n^2\pi^2}\right)\right]_0^{40}$$

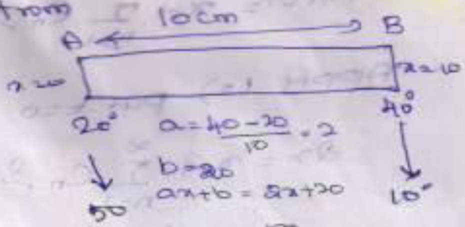
$$= \frac{1}{20}\left[-\frac{x\,40}{n\pi}\cos\left(\frac{n\pi x}{40}\right) + \frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)\right]_0^{40}$$

$$= \frac{1}{20}\left[-\frac{1600}{n\pi}(-1)^n\right] = -\frac{80(-1)^n}{n\pi}$$

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{-80(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

A metal bar 10cm long with insulated sides has its ends A and B kept at 20°c and 40°c resp. Until steady state cond. prevail. The temp. at A is then suddenly raised to 50°c and at the same instant that at B is lowered to 10°c. Find the sub.seq. temp. at any pt. of the bar at any time.

Sol: The temp. $u(x,t)$ is from




$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$a = \frac{40-20}{10} = 2$

$b = 20$

$ax+b = 2x+20$

The cond. are

① $u(0,t) = 50$

② $u(10,t) = 10$

③ $u(x,0) = 2x+20$

$a = \frac{10-50}{10} = -4$

$b = 50$

$ax+b = -4x+50$

The suit. sol. is

$$u(x,t) = [A\cos px + B\sin px]\, C e^{-\alpha^2 p^2 t} - 4x + 50 \quad \text{--- ①}$$

Apply ① [Put $x=0$]

$$50 = A(Ce^{-\alpha^2 p^2 t}) + 50$$

$$A(Ce^{-\alpha^2 p^2 t}) = 0 \Rightarrow \boxed{A=0}$$

Apply ② Put $x=10$

$$10 = B\sin(10p)\, Ce^{-\alpha^2 p^2 t} - 40 + 50$$

$$0 = B\sin(10p)\, Ce^{-\alpha^2 p^2 t}$$

$$B\sin 10p = 0 = \sin n\pi \qquad 10p = n\pi \qquad p = \frac{n\pi}{10}$$

∴ ① Becomes

$$u(x,t) = B\sin\left(\frac{n\pi x}{10}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} - 4x + 50$$

Most gen. sol.

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} - 4x + 50 \quad \text{--- ②}$$

Apply ③ Put [t=0]

$$2x+20 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{10}\right) - 4x + 50$$

$$6x - 30 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{10}\right)$$

$$C_n = \frac{2}{10}\int_0^{10} (6x-30)\sin\left(\frac{n\pi x}{10}\right) dx.$$

$$= \frac{1}{5}\left[-\frac{(6x-30)\,10}{n\pi}\cos\left(\frac{n\pi x}{10}\right) + \frac{600}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[-\frac{300}{n\pi}(-1)^n - \frac{300}{n\pi}(1)\right]$$

$$= \frac{1}{5} \times -\frac{300}{n\pi}\left[(-1)^n + 1\right]$$

$$= -\frac{60}{n\pi}\left[(-1)^n + 1\right]$$

$$= \begin{cases} -\dfrac{120}{n\pi} & n = 2, 4, 6 \ldots \\ 0 & n = 1, 3, 5 \ldots \end{cases}$$

② becomes

$$\therefore u(x,t) = \sum_{n=2}^{\infty} -\frac{120}{n\pi} \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} - 4x + 50$$

— x —

Two dimensional Heat eqn.

The two dimensional heat eqn. is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ — ①}$$

The temp. fn. is $u(x,y)$

In ① $A=1$ $B=0$ $C=1$

$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0.$

Two dimensional heat eqn. is elliptic

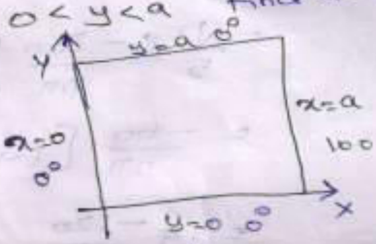
Write down all possible sol. of two dimensional heat eqn.

① $u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

② $u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$

③ $u(x,y) = (Ax+B)(Cy+D)$

A Square ~~Pate~~ plate is bounded by the lines $x=0$ $y=0$ $x=a$ $y=a$. The lines $x=0$ $y=0$ $y=a$ are kept at $0°C$. The Side $x=a$ is kept at temp. gn. by $u(a,y)=100$ $0<y<a$ Find $u(x,y)$




The temp $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Condn. are

① $u(x,0)=0$

② $u(x,a)=0$

③ $u(0,y)=0$

④ $u(a,y)=100$

The Suitable Sol. is

$u(x,t) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$ — ⓘ

Apply ①

$0 = (Ae^{px} + Be^{-px})(C)$

$\Rightarrow \boxed{C=0}$

~~Apply ②~~ Sub $C=0$ in I

$u(x,t) = (Ae^{px} + Be^{-px})(D\sin py)$ — II

Apply ② in II

$\ldots -px)(D\sin pa) \ldots$ — ③

$D \sin pa = 0$

$\sin pa = 0 = \sin n\pi$

$\boxed{p = n\pi/a}$

Sub $p = n\pi/a$ in (ii)

$u(x,y) = \left(Ae^{\frac{n\pi x}{a}} + Be^{-\frac{n\pi x}{a}}\right)\left(D \sin \frac{n\pi y}{a}\right)$ — (iii)

Apply (iii)

$u(0,y) = (A+B)\left(D \sin \frac{n\pi y}{a}\right) = 0$

$B = -A$

$\therefore$ (3) Becomes

$u(x,y) = A\left(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}\right) D \sin\left(\frac{n\pi y}{a}\right)$ — (iv)

The most gen. sol. is

$u(x,y) = \sum_{n=1}^{\infty} D_n\left(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}\right) \sin\left(\frac{n\pi y}{a}\right)$ — (v)

Apply (4) in (v)

$u(a,y) = \sum_{n=1}^{\infty} D_n\left(e^{n\pi} - e^{-n\pi}\right) \sin\left(\frac{n\pi y}{a}\right) = 100$

$D_n\left(e^{n\pi} - e^{-n\pi}\right) = \frac{2}{a}\int_0^a 100 \sin\left(\frac{n\pi y}{a}\right) dy$

$= \frac{200}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$

$= \frac{200}{a}\left[-\cos\left(\frac{n\pi y}{a}\right)\left(\frac{a}{n\pi}\right)\right]_0^a$

$= \frac{200}{a}\left[-\frac{a}{n\pi}(-1)^n + \frac{a}{n\pi}\right]$

$= \frac{200}{a} \times \frac{a}{n\pi}\left[-(-1)^n + 1\right]$

$D_n\left[e^{n\pi} - e^{-n\pi}\right] = \begin{cases} 0 & n = 2,4,6 \ldots \\ \frac{400}{n\pi} & n = 1,3,5 \ldots \end{cases}$

$$D_n = \frac{400}{n\pi\left[e^{n\pi}-e^{-n\pi}\right]} \quad n = 1,3,5 \ldots$$

$$u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{400}{n\pi\left[e^{n\pi}-e^{-n\pi}\right]}\left[e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}}\right] \sin\left(\frac{n\pi y}{a}\right)$$

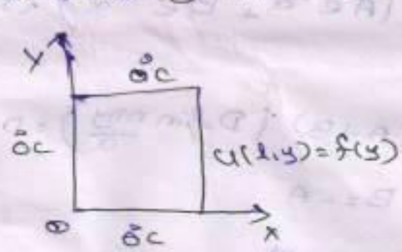
Finite Plate with Value given in y-direction.

(1) $u(x,0)=0$

(2) $u(x,l)=0$

(3) $u(0,y)=0$

(4) $u(l,y)=f(y)$




The Suitable Sol.

$$u(x,y) = (Ae^{px}+Be^{-px})(C\cos py + D\sin py)$$

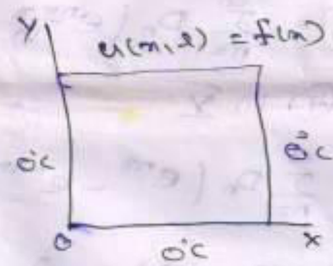
x- Direction

(1) $u(0,y)=0$

(2) $u(l,y)=0$

(3) $u(x,0)=0$

(4) $u(x,l)=f(x)$



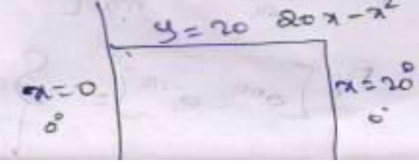

The Suitable Sol.

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py}+De^{-py})$$

—x—

A Sq. plate is bounded by the lines $x=0, y=0$, $x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is given by $u(x,20) = x(20-x) = 20x-x^2$, $0<x<20$ while the other edges are kept at 0°C. Find the steady state temp. distn. in the plate.

Sol:



35, Vinitha

The temp. $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The cond. are

(1) $u(0,y)=0$ (2) $u(20,y)=0$ (3) $u(x,0)=0$ (4) $u(x,20) = 20x - x^2$

The suitable sol. is

$$u(x,y) = [A\cos px + B\sin px][Ce^{py} + De^{-py}]$$

Apply (1) $0 = A(Ce^{py} + De^{-py}) \Rightarrow \boxed{A=0}$

Apply (2) $0 = B\sin(20p)\,(Ce^{py} + De^{-py})$

$B\sin(20p) = 0 \quad 20p = n\pi \Rightarrow p = n\pi/20$

Apply (3)

$0 = B\sin px\,(C+D)$

$C + D = 0 \Rightarrow D = -C$

$\therefore$ (1) Becomes

$$u(x,y) = B\sin px\, C\cdot[e^{py} - e^{-py}]$$

$$= B\sin\left(\frac{n\pi x}{20}\right) C\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

Most gen. sol.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right)\left[e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right]$$

Apply (4)

$$20x - x^2 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right)\left[e^{n\pi} - e^{-n\pi}\right]$$

$$C_n\left[e^{n\pi} - e^{-n\pi}\right] = \frac{2}{20}\int_0^{20} (20x - x^2)\sin\left(\frac{n\pi x}{20}\right)dx.$$

$$= \frac{1}{10}\left[\frac{-20(20x - x^2)}{n\pi}\cos\left(\frac{n\pi x}{20}\right) + \frac{400}{n^2\pi^2}(20 - 2x)\sin\left(\frac{n\pi x}{20}\right) - \frac{16000}{n^3\pi^3}\cos\left(\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[\frac{-16000}{n^3\pi^3}(-1)^n + \frac{16000}{n^3\pi^3}(1)\right]$$

$$= \frac{1600}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

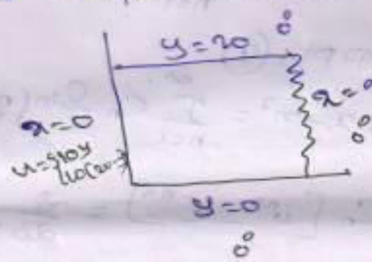
$$= \begin{cases} 0 & n = 2,4,6 \ldots \\ \frac{3200}{n^3\pi^3} & n = 1,3,5 \ldots \end{cases}$$

$$C_n = \frac{3200}{n^3\pi^3\left(e^{n\pi} - e^{-n\pi}\right)}$$

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3\left(e^{n\pi} - e^{-n\pi}\right)} \sin\left(\frac{n\pi x}{20}\right)\left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}\right)$$

—x—

A rectangular plate with insulated surfaces is 20cm wide and so long compared to its width that it may be considered infinite in length. If the temp. at the short edge $x=0$ is $u = \begin{cases} 10y & 0 \le y \le 10 \\ 10(20-y) & 10 \le y \le 20. \end{cases}$ and the two long edges as well as the other short edge are kept at 0°C find the steady state temp. distribution in the plate.




Sol:
The temp $u(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The cond. are.

1. $u(x,0) = 0$
2. $u(x,20) = 0$
3. $u(\infty,y) = 0$
4. $u(0,y) = u = \begin{cases} 10y & 0 \le y \le 10 \\ 10(20-y) & 10 \le y \le 20. \end{cases}$

The suitable soln. is

$u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$ — (1)

Apply (1) :

$0 = (Ae^{px} + Be^{-px})\, C$

$C = 0$

Apply (2) .

$0 = (Ae^{px} + Be^{-px})(D\sin 20p)$

$\sin 20p = 0 = \sin n\pi$

$p = n\pi/20$

Apply (3) $x = \infty$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$0 = [Ae^{\infty} + Be^{-\infty}]\, D\sin py$

$= (A.\infty)(D\sin py) = 0$

$\boxed{A = 0}$

$\therefore$ (1) Becomes.

$u(x,y) = Be^{-\frac{n\pi x}{20}} D\sin\left(\frac{n\pi y}{20}\right)$

Most genr. sol..

$u(x,y) = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi x}{20}} \sin\left(\frac{n\pi y}{20}\right)$

Apply (4)

$u = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{20}\right)$

$D_n = \frac{2}{20}\int_0^{20} u \sin\left(\frac{n\pi y}{20}\right) dy$

$= \frac{1}{10}\left[\int_0^{10} 10y \sin\left(\frac{n\pi y}{20}\right) dy + \int_{10}^{20} 10(20-y)\sin\left(\frac{n\pi y}{20}\right) dy\right]$

$= \left[y\left(-\cos\left(\frac{n\pi y}{20}\right)\left(\frac{20}{n\pi}\right)\right) + \sin\left(\frac{n\pi y}{20}\right)\left(\frac{400}{n^2\pi^2}\right)\right]_0^{20} +$

$\left[(20-y)\left(-\cos\left(\frac{n\pi y}{20}\right)\left(\frac{20}{n\pi}\right)\right) - \sin\left(\frac{n\pi y}{20}\right)\left(\frac{400}{n^2\pi^2}\right)\right]_0$

$$= \left[ \frac{-200}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{400}{n^2\pi^2} \sin(n\pi/2) + \right.$$

$$\left. \frac{200}{n\pi} \cos\left[n\pi/2\right) + \frac{400}{n^2\pi^2} \sin\left(n\pi/2\right) \right]$$

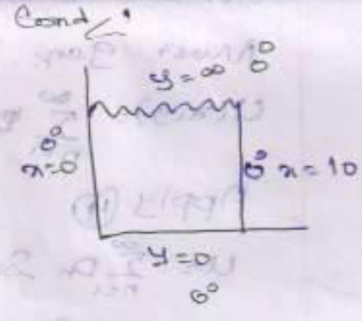
$$D_n = \frac{800}{n^2\pi^2} \sin\left(n\pi/2\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2} \sin\left(n\pi/2\right) e^{-\frac{n\pi x}{20}} \sin\left(\frac{n\pi y}{20}\right)$$

—x—

A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considering in finite in length. The temp at short edge y=0 is gn by $u = \begin{cases} 20x & 0 \le x \le 5 \\ 20(10-x) & 5 \le x \le 10 \end{cases}$ all the other edges are kept at 0°c.

Find the steady state cond:




The temp u(x,y) is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The cond are.

1. $u(0,y) = 0$
2. $u(10,y) = 0$
3. $u(x,\infty) = 0$
4. $u(x,0) = u = \begin{cases} 20x & 0 \le x \le 5 \\ 20(10-x) & 5 \le x \le 10. \end{cases}$

The suitable sol is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \text{—} I$$

Apply, x=0 ① cond

$$= A(Ce^{py} + De^{-py})$$

$\boxed{A=0}$

Apply ② Cond.. $x=10$

$$0 = B\sin(10p)\left(Ce^{py}+De^{-py}\right)$$

$$B\sin 10p = 0$$

$$\sin 10p = 0 = \sin n\pi$$

$$p = n\pi/10$$

Apply ③

$$0 = B\sin(px)\left(Ce^{\infty}+De^{-\infty}\right)$$

$$0 = B\sin px\, C(\infty)$$

$\boxed{C=0}$

① Becomes.

$$u(x,y) = B\sin\left(\frac{n\pi x}{10}\right)De^{-\frac{n\pi y}{10}}$$

The most gen. soln is

$$u(x,y) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n\pi y}{10}}$$

Apply 4

$$u = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{10}\right)$$

$$D_n = \frac{2}{10}\left[\int_0^{5} 20x \sin\left(\frac{n\pi x}{10}\right) + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= 4\left[\left[\frac{-10x}{n\pi}\cos\left(\frac{n\pi x}{10}\right)+\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)\right]_0^5 + \left[\frac{-10(10-x)}{n\pi}\cos\left(\frac{n\pi x}{10}\right)-\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)\right]_5^{10}\right]$$

$$= 4\left(-\frac{50}{n\pi}\cos(n\pi/2)+\frac{100}{n^2\pi^2}\sin(n\pi/2)+\frac{50}{n\pi}\cos(n\pi/2)+\frac{100}{n^2\pi^2}\sin(n\pi/2)\right)$$

$$D_n = \frac{800}{n^2\pi^2}\sin(n\pi/2)$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}}.$$

Problems on Polar Co-ordinates.

A thin Semi circular plate of radius 'a' has its bounding diameter kept at temp. zero and its circumference at k. Find the temp distribution in the steady state.

Soln Let the Centre of the Circle be pole and the bounding diameter as the initial line.

The steady state temp at any pt $p(r,\theta)$ be $u(r,\theta)$ then $u$ satisfies the eqn.

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

The Condn (1) $u(r,0) = 0$ in $0 \le r \le a$

(2) $u(r,\pi) = 0$ $0 \le r \le a$

(3) $u(a,\theta) = k$.

the suitable soln

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \text{—(I)}$$

Apply (1) cond in I

$$u(r,0) = (c_1 r^p + c_2 r^{-p}) c_3 = 0$$

$$(c_1 r^p + c_2 r^{-p}) \neq 0.$$

$$\Rightarrow c_3 = 0$$

Sub $c_3 = 0$ in eqn (I)

$$u = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta \quad \text{— (II)}$$

$u(r,\pi) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi = 0$

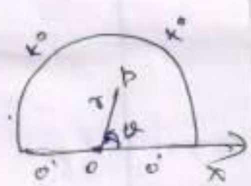
$c_1 r^p + c_2 r^{-p} \neq 0, \quad c_4 \neq 0$

$\sin p\pi = 0 = \sin n\pi \quad \boxed{p = n}$

Sub $p = n$ in ② $-$ ③

$u = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta$

$u = 0 \quad r = 0 \quad \therefore \boxed{c_2 = 0}$



③ $\Rightarrow u = c_1 r^n c_4 \sin n\theta$.

$u = C_n r^n \sin n\theta$.

The most gen. soln

$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^n \sin n\theta$. ——— IV

Apply Cond. ③ in IV

$u(a,\theta) = \sum_{n=1}^{\infty} C_n a^n \sin n\theta = k$.

$\Rightarrow \sum b_n \sin n\theta = k$ where $b_n = C_n a^n$

$b_n = \frac{2}{\pi} \int_0^{\pi} k \sin n\theta \, d\theta$.

$= \frac{2k}{n\pi} \left[ -\cos n\theta \right] = \frac{-2k}{n\pi} \left[ \cos n\pi - 1 \right]$

$= \frac{2k}{n\pi} \left[ 1 - \cos n\pi \right]$

$C_n = \frac{1}{a^n} b_n = \frac{1}{a^n} \frac{2k}{n\pi} (1 - \cos n\pi)$

$C_n = \begin{cases} 0 & n = \text{even} \\ \frac{4k}{a^n n\pi} & n = \text{odd}. \end{cases}$

$u(r,\theta) = \frac{4k}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{r^n}{a^n n} \sin n\theta$