Unit - I
Partial Differential Equations

Formation of Partial Differential equations by Elimination of arbitrary constants

Consider an equation $f(x, y, z, a, b) = 0$ → (1)

where a & b denote arbitrary constants.

A p.d.e is formed by eliminating the arbitrary constants that occur in the functional relation between the variables.

Using $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

---

1. form the p.d.e by eliminating the arbitrary constants a & b form $z = ax + by$ → (1)

Diff p.w.r.to x we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a.$$

Diff p.w.r.to y we get

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b.$$

∴ Eqn (1) becomes, $z = px + qy$.

---

2. Eliminate the arbitrary constants a & b form $z = (x^2 + a)(y^2 + b)$.

Sol:

$$z = (x^2 + a)(y^2 + b)$$

Diff P. w. r to x.

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b) \Rightarrow \frac{p}{2x} = y^2 + b$$

Diff p. w. r to y.

$$q = 2y(x^2 + a) \Rightarrow \frac{q}{2y} = x^2 + a$$

∴ Eqn ① becomes

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$4xyz = pq.$$

3. $z = a(x + y) + b$

Sol:

$$z = a(x + y) + b$$

Diff w.r.to x,

$$p = a \rightarrow ①$$

Diff w.r.to y,

$$q = a \rightarrow ②$$

From ① & ② we get $pq = a$. $p = q$.

4. Find the partial differential equation of all planes having equal intercepts on the x & y axis.

Sol: Intercept form of the plane equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Given $a = b$ (Equal intercepts on the $x \& y$ axis)

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1.$$

Diff. w.r to x, we get

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial x} = 0.$$

$$\Rightarrow \frac{1}{a} = -\frac{1}{c} \frac{\partial z}{\partial x} = -\frac{1}{c} p \longrightarrow ①$$

Diff. w.r. to y, we get

$$\frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{1}{a} = -\frac{1}{c} \frac{\partial z}{\partial y} = -\frac{1}{c} q \longrightarrow ②$$

From ① & ②,

$$-\frac{1}{c} p = -\frac{1}{c} q$$

$$p = q.$$

6. $(x-a)^2 + (y-b)^2 + z^2 = 1.$

Sol:

Diff w.r. to x, we get

$$2(x-a) + 2z \frac{\partial z}{\partial x} = 0$$

$$2(x-a) = -2zp$$

$$x-a = -zp \rightarrow ①$$

Diff p.w.r.to y, we get

$$2(y-b) + 2z \frac{\partial z}{\partial y} = 0$$

$$2(y-b) = -2zq$$

$$y-b = -zq \rightarrow ②$$

Using ① & ② , we get

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2 p^2 + z^2 q^2 + z^2 = 1$$

$$z^2(p^2 + q^2 + 1) = 1.$$

⑥ $Z = (x+a)^3 + (y-b)^2$

<u>Sol:</u>

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = 3(x+a)^2 . 1$$

$$\Rightarrow \frac{p}{3} = (x+a)^2$$

Diff p.w.r to y, we get

$$\frac{\partial z}{\partial y} = 2(y-b)$$

$$\frac{q}{2} = y-b.$$

$$z = \left(\frac{p}{3}\right)^{3/2} + \left(\frac{q}{2}\right)^2 \qquad z = \left(\frac{p}{3}\sqrt{\frac{p}{3}}\right)^3 + \left(\frac{q}{2}\right)^2$$

$$z = \left(\frac{p^{3/2}}{3^{3/2}}\right)^3 + \left(\frac{q}{2}\right)^2.$$

(7) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha.$

Sol:

$$(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$$

Diff w.r. to x we get

$$2(x-a) = 2 zp \cot^2 \alpha$$

$$x-a = zp \cot^2 \alpha$$

Diff w.r. to y, we get

$$2(y-b) = 2zq \cot^2 \alpha$$

$$y-b = zq \cot^2 \alpha$$

$$z^2 p^2 \cot^4 \alpha + z^2 q^2 \cot^4 \alpha = z^2 \cot^2 \alpha$$

$\div z^2 \cot^2 \alpha, \quad p^2 \cot^2 \alpha + q^2 \cot^2 \alpha = 1.$

Formation of PDE by Elimination of arbitrary constants Functions:

Formation of PDE by elimination of arbitrary functions from the elimination of one arbitrary functions from a given relation gives a PDE of first order while elimination of two arbitrary function from a given relation gives a second order or higher order PDE.

Using $p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2},$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

$$\phi(u,v)=0 \Rightarrow \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\[2mm] \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0.$$

① Eliminate the arbitrary function f from $z = f\left(\dfrac{y}{x}\right)$ form a PDE.

Sol:

Given $z = f\left(\dfrac{y}{x}\right)$

Diff p.w. r.to x, we get

$$\dfrac{\partial z}{\partial x} = f'\left(\dfrac{y}{x}\right) \cdot \dfrac{-y}{x^2}$$

$$\Rightarrow p = f'\left(\dfrac{y}{x}\right) \cdot \dfrac{-y}{x^2} \rightarrow ①$$

Diff p.w. r.to y, we get

$$\dfrac{\partial z}{\partial y} = f'\left(\dfrac{y}{x}\right) \cdot \dfrac{1}{x}$$

$$\Rightarrow q = f'\left(\dfrac{y}{x}\right) \cdot \dfrac{1}{x} \rightarrow ②$$

from ① & ②.

$$\dfrac{p}{q} = \dfrac{-y}{x^2} \cdot \dfrac{x}{1}$$

$$\dfrac{p}{q} = \dfrac{-y}{x}$$

$$px = -qy$$

$$\Rightarrow px + qy = 0.$$

② $z = f(xy)$

Sol:
$z = f(xy)$

Diff w.r to x, we get

$\dfrac{\partial z}{\partial x} = f'(xy) \cdot y$

$p = f'(xy) \cdot y$

Diff w.r to y, we get

$\dfrac{\partial z}{\partial y} = f'(xy) \cdot x$

$q = f'(xy) \cdot x$

$\dfrac{p}{q} = \dfrac{y}{x}$

$px = qy$

$\Rightarrow px - qy = 0.$

③ $\phi\left(z^2 - xy, \dfrac{x}{z}\right) = 0.$

Sol:
Here $u = z^2 - xy$, $\quad v = \dfrac{x}{z}$

$\dfrac{\partial u}{\partial x} = 2z\,\dfrac{\partial z}{\partial x} - y$ $\qquad$ $\dfrac{\partial v}{\partial x} = \dfrac{z \cdot 1 - x \cdot \frac{\partial z}{\partial x}}{z^2}$

$\qquad = 2zp - y$ $\qquad\qquad\qquad = \dfrac{z - xp}{z^2}$

$\dfrac{\partial u}{\partial y} = 2z\,\dfrac{\partial z}{\partial y} - x$ $\qquad$ $\dfrac{\partial v}{\partial y} = \dfrac{z \cdot 0 - x \cdot q}{z^2} = \dfrac{-xq}{z^2}$

$\qquad = 2zq - x$

$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\[2mm] \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial y} \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 2zp-y & \dfrac{z-xp}{z^2} \\ 2zq-x & -\dfrac{xq}{z^2} \end{vmatrix} = 0$$

$$(2zp-y)\left(-\dfrac{xq}{z^2}\right) - (2zq-x)\left(\dfrac{z-xp}{z^2}\right) = 0$$

$$\dfrac{1}{z^2}\left[-2zpxq + yxq - 2zqz + 2zqxp + xz - x^2p\right] = 0$$

$$xyq + xz - 2z^2q - x^2p = 0$$

$$xz = 2z^2q + x^2p - xyq$$

$$xz = x^2p \bar{=} q(xy - 2z^2)$$

4. $z = f(x^2+y^2)$.

Sol:

$$z = f(x^2+y^2)$$

Diff p.w.r.to x, we get

$$\dfrac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

$$p = 2x \, f'(x^2+y^2)$$

Diff p.w.r.to y, we get

$$\dfrac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y$$

$$q = 2y \, f'(x^2+y^2)$$

$$\dfrac{P}{q} = \dfrac{2x \, f'(x^2+y^2)}{2y f'(x^2+y^2)}$$

$$\dfrac{p}{q} = \dfrac{x}{y}$$

$$py - qx = 0.$$

5. $z = xf(2x+y) + g(2x+y)$

Sol:

$$z = xf(2x+y) + g(2x+y)$$

Diff p. w. r. to x.

$$p = \frac{\partial z}{\partial x} = f(2x+y) + x f'(2x+y) \cdot 2 + g'(2x+y) \cdot 2 \quad \rightarrow \text{①}$$

$$q = x f'(2x+y) + g'(2x+y) \quad \rightarrow \text{②}$$

$$r = f'(2x+y) \cdot 2 + x f''(2x+y) \cdot 4 + f'(2x+y) \cdot 2$$

$$+ g''(2x+y) \cdot 4 \quad \rightarrow \text{③}$$

$$t = x f''(2x+y) + g''(2x+y) \quad \rightarrow \text{④}$$

$$s = f'(2x+y) + 2x f''(2x+y) + 2g''(2x+y) \quad \rightarrow \text{⑤}$$

$\text{③} \Rightarrow \ r = 4 f'(2x+y) + 4\left[x f''(2x+y) + g''(2x+y) \right]$

$$r = 4 f'(2x+y) + 4t \quad \rightarrow \text{⑥}$$

$\text{⑤} \Rightarrow \ s = 2\left[x f''(2x+y) + g''(2x+y) \right] + x f''(2x+y)$

$$s = 2t + x f'(2x+y) \quad \rightarrow \text{⑦}$$

$\text{⑥} - 4 \times \text{⑦}$

$$r = 4f'(2x+y) + 4t$$
$$4s = 4f'(2x+y) + 8t$$
$$\underline{(-) \quad\quad (-) \quad\quad\quad (-)}$$

$$r - 4s = -4t$$

$$r - 4s + 4t = 0.$$

⑥ From the P. de by eliminating ϕ from

$$\phi(x^2+y^2+z^2, \ ax+by+cz)=0.$$

Sol: Here $u=x^2+y^2+z^2$ $v=ax+by+cz$.

$$\frac{\partial u}{\partial x}=2x+2zp \qquad\qquad \frac{\partial v}{\partial x}=a+cp$$

$$\frac{\partial u}{\partial y}=2y+2zq \qquad\qquad \frac{\partial v}{\partial y}=b+cq.$$

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\[2mm] \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}=0$$

$$\begin{vmatrix} 2x+2zp & a+cp \\[2mm] 2y+2zq & b+cq \end{vmatrix}=0$$

$$2(x+zp)(b+cq)-2(y+zq)(a+cp)=0.$$

Singular Integrals:-

Problems based on solution. of P.DE in ordinary cases:

1. Solve: $\frac{\partial z}{\partial x}=\sin x$.

sol: diff w.r.to x.

Integrating p.w. r.to x.

$$z=-\cos x + f(y).$$

$f(y)$ is arbitrary constant.

2. Solve: $\dfrac{\partial^2 z}{\partial x^2} = xy.$

Sol:

$$\dfrac{\partial^2 z}{\partial x^2} = xy.$$

ie, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = xy$

Integrating p.w.r. to x,

$$\dfrac{\partial z}{\partial x} = \dfrac{x^2}{2}y + f(y)$$

Integrating p.w.r. to x

$$z = \dfrac{x^3}{3 \cdot 2}y + xf(y) + F(y).$$

where both $f(y)$ & $F(y)$ are arbitary.

8. Solve: $\dfrac{\partial z}{\partial x} = 3x - y, \qquad \dfrac{\partial z}{\partial y} = -x + \cos y.$

Sol:

$$\dfrac{\partial z}{\partial x} = 3x - y.$$

Integrating p.w.r. to x,

$$z = 3\dfrac{x^2}{2} - xy + f(y)$$

$$\dfrac{\partial z}{\partial y} = -x + \cos y$$

Integrating p.w.r. to y,

$$z = -xy + \sin y + c.$$

$$xz = \dfrac{3}{2}x^2 - xy + \sin y + c.$$

4. Solve: $\dfrac{\partial^2 z}{\partial x^2} = \sin y$

Sol: $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \sin y$

Integrating p.w.r.to x,

$$\dfrac{\partial z}{\partial x} = x \sin y + f(y)$$

Again integrating p.w.r. to x,

$$z = \dfrac{x^2}{2} \sin y + x \, f(y) + F(y)$$

where both $f(y) \, \& \, F(y)$ are arbitrary.

5. Solve: $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that $x = 0$,

$\dfrac{\partial z}{\partial x} = a \sin y$, and $\dfrac{\partial z}{\partial y} = 0$.

Sol: Given $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$

$$\dfrac{\partial^2 z}{\partial x^2} - a^2 z = 0.$$

The A.E is $m^2 + a^2 = 0$

$$m^2 = a^2$$

$$m = \pm ia$$

$$z = A e^{ax} + B e^{-ax}.$$

Since z is a function of x $\&y$.
Therefore $A \& B$ will be the functions of y alone. Hence

$$z = f(y) e^{ax} + \varphi(y) e^{-ax} \rightarrow ①$$

where $f(y) \& \varphi(y)$ are functions of y alone.

Diff w.r. to x,

$$\frac{\partial z}{\partial x} = p = a f(y) e^{ax} - a \varphi(y) e^{-ax} \rightarrow ②$$

$$\frac{\partial z}{\partial y} = q = f'(y) e^{ax} + \varphi'(y) e^{-ax} \rightarrow ③$$

Case: (i)   Given $\frac{\partial z}{\partial x} = a\sin y$ when $z = 0$.

$$② \Rightarrow a\sin y = a f(y) - a\varphi(y)$$

$$a\sin y = a[f(y) - \varphi(y)]$$

$$\sin y = f(y) - \varphi(y) \rightarrow ④$$

case: (ii)   Given $\frac{\partial z}{\partial y} = 0$ when $x = 0$.

$$③ \Rightarrow 0 = f'(y) + \varphi'(y).$$

Integrating, we get

$$f(y) + \varphi(y) = k \rightarrow ⑤$$

$$④ + ⑤ \Rightarrow f(y) = \frac{1}{2}(\sin y + k)$$

⑤-④ ⇒ $\phi(y) = \frac{1}{2}(k - \sin y)$

$$z = \frac{1}{2}(\sin y + k)e^{ax} + \frac{1}{2}(k - \sin y)\,e^{-ax}$$

$$= \frac{1}{2}\sin y\, e^{ax} + \frac{1}{2}k e^{ax} + \frac{1}{2}k e^{-ax} - \frac{1}{2}\sin y\, e^{-ax}$$

$$= \frac{1}{2}\sin y\,(e^{ax} - e^{-ax}) + \frac{1}{2}k(e^{ax} + e^{-ax})$$

$$= \sin y\left(\frac{e^{ax} - e^{-ax}}{2}\right) + k\left(\frac{e^{ax} + e^{-ax}}{2}\right)$$

$$z = \sin y\,\sinh ax + k\,\cosh ax.$$

Define : Singular Integral

Let $f(x, y, z, p, q) = 0.$ →①

Let the complete integral be

$$\phi(x, y, z, a, b) = 0 \quad →②$$

Diff ② p.w.r. to a & b in turn we get

$$\frac{\partial \phi}{\partial a} = 0 \quad →③ \text{ and}$$

$$\frac{\partial \phi}{\partial b} = 0 \quad →④$$

The elimination of a & b from the three equations ②,③ & ④ if it exists, is called the singular Integral.

Type:1 $f(p,q)=0$

[The equations contain p and q only]

Suppose that $z = ax + by + c$ is a trial solution of $f(p,q)=0$.

where $p=a$, $q=b$ we get $f(a,b)=0$.

Here a & b are the constant.

Eliminate any one constant we get the complete solution.

1. Find the complete solution of $\sqrt{p} + \sqrt{q} = 1$

Sol:

Given $\sqrt{p} + \sqrt{q} = 1$. $\longrightarrow$①

This equation of the form $f(p,q)=0$.

Hence the trial solution is $z = ax + by + c \longrightarrow$②

where $p=a$ & $q=b$.

Substitute in eqn ① we get

$$\sqrt{a} + \sqrt{b} = 1$$
$$\Rightarrow \sqrt{b} = 1 - \sqrt{a} \Rightarrow \sqrt{b} = (1-\sqrt{a})^2$$

$\therefore z = ax + (1-\sqrt{a})^2 y + c.$

2. $p + q = pq.$

Sol: Given $p + q = pq \longrightarrow$①

This equation of the form $f(p,q)=0$

Hence the trial solution is $z = ax + by + c \longrightarrow$②

where $p = a$ & $q = b$

Substitute in eqn ①, we get

$$a + b = ab$$
$$\Rightarrow b - ab - a$$
$$b - ab = a$$
$$\Rightarrow b(1-a) = a$$
$$b = \frac{a}{1-a}$$

The complete solution is $z = ax + \left(\frac{a}{1-a}\right) y + c$

⑧ $p^2 + q^2 = npq$.

Sol: Given $p^2 + q^2 = npq$.

This eqn is of the form $z = ax + f(p, q) = 0$

Hence the trial solution is $z = ax + by + c$

where $p = a$ & $q = b$

$$a^2 + b^2 = nab$$
$$b^2 - nab + a^2 = 0$$
$$b = \frac{na \pm \sqrt{a^2 n^2 - 4a^2}}{2}$$
$$= \frac{a}{2} \left[n \pm \sqrt{n^2 - 4} \right]$$

The complete solution is

$$z = ax + \frac{a}{2} \left[n + \sqrt{n^2 - 4} \right] y + c.$$

(4) $p - 3q = 6$.

Sol: Given $p - 3q = 6$

This eqn of the form $f(p, q) = 0$

Hence the trial solution is $z = ax + by + c$

where $p = a$ & $q = b$

$a - 3b = 6$.

$\Rightarrow -3b = 6 - a$

$\Rightarrow b = \dfrac{6 - a}{-3} = -2 + \dfrac{a}{3}$.

The complete solution is

$$z = ax + \left(-2 + \dfrac{a}{3}\right) y + c.$$

(5) $p - q = 0$.

Sol:

Given $p - q = 0$.

This eqn of the form $f(p, q) = 0$

Hence the trial solution is $z = ax + by + c \rightarrow$ ②

Sub. ② in ①. Here $p = a$ & $q = b$

$a - b = 0$.

$b = a$

The complete solution is

$$z = ax + ay + c = a(x + y) + c$$

Type : 2 Clairaut's form

$$z = px + qy + f(p,q).$$

This eqn of the form $z = px + qy + f(p,q)$.

∴ The complete integral is

$$z = ax + by + f(a,b).$$

To find the singular integral

Diff p.w. r. to a & b,

we get the solution in terms of x, y, z.

To find the general solution

Put $b = f(a)$

Eliminate 'a' we get the general solution.

1. Solve : $z = px + qy + pq$.

Sol: Given $z = px + qy + pq.$ → ①

This eqn is of the form $z = px + qy + f(p,q)$ → ②

∴ The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by + ab.$$

To find singular integral

Diff p.w.r.to a & b.

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + b = 0$$

$$\Rightarrow b = -x$$

$$\frac{\partial z}{\partial b} = 0 \implies y + a = 0$$

$$\implies a = -y.$$

$$\therefore z = (-y)x + (-x)y + (-y)(-x)$$

$$= -xy - xy + xy$$

$$z = -xy$$

$$z + xy = 0.$$

which is a singular solution.

To get the general integral

put $b = f(a)$ in eqn ①.

$$z = ax + f(a)y + af(a) \longrightarrow ④$$

Diffp. w.r.to a, $\frac{\partial z}{\partial a} = 0.$

$$\implies x + f'(a)y + a f'(a) + f(a) = 0 \longrightarrow ⑤$$

Eliminate 'a' between ④ & ⑤ we get the general solution.

② $z = px + qy + p^2 - q^2$

sol:

Given $z = px + qy + p^2 - q^2 \longrightarrow ①$

This eqn of the form $z = px + qy + f(p,q) ②$

The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by + a^2 - b^2 \longrightarrow ③$$

To find singular integral

Diff p.w.r to a & b,

$\dfrac{\partial z}{\partial a} = 0 \Rightarrow x + 2a = 0$

$\qquad\qquad 2a = -x$

$\qquad\qquad a = \dfrac{-x}{2}$

$\dfrac{\partial z}{\partial b} = 0 \Rightarrow y - 2b = 0,$

$\qquad\qquad \Rightarrow y = 2b$

$\qquad\qquad \Rightarrow b = \dfrac{y}{2}.$

Sub a, b in ②,

$z = -\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{x^2}{4} - \dfrac{y^2}{4}$

$\quad = \dfrac{-2x^2 + 2y^2 + x^2 - y^2}{4}$

$z. = \dfrac{-x^2 + y^2}{4},$

$4z = y^2 - x^2$ is the singular integral

To find the general integral

$\qquad$ Put $b = f(a)$ in ②

$\qquad z = ax + f(a) y + a^2 - (f(a))^2 \to ④$

$\qquad \dfrac{\partial z}{\partial a} = 0$

$\qquad \Rightarrow x + f'(a) y + 2a - 2 f(a). f'(a) = 0 \to ⑤$

Eliminate 'a' between ④ & ⑤ we get

⑧ Solve: $z = px + qy + \sqrt{p^2 + q^2 + 1}$ ⑤

Sol:

Given $z = px + qy + \sqrt{p^2 + q^2 + 1}$

This eqn is of the form $z = px + qy + f(p, q)$

∴ The complete integral is

$$z = ax + by + f(a, b)$$

(ie) $z = ax + by + \sqrt{a^2 + b^2 + 1}$ → ①

To find : singular integral

Diff p. w. r. to a & b,

$$\frac{\partial z}{\partial a} = a \Rightarrow x + \frac{1}{2}(a^2 + b^2 + 1)^{-\frac{1}{2}} \cdot 2a = 0$$

$$\Rightarrow x + \frac{a}{\sqrt{a^2 + b^2 + 1}} = 0$$

ii.

$$x = \frac{-a}{\sqrt{a^2 + b^2 + 1}} \quad —②$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow y + \frac{1}{2}(a^2 + b^2 + 1)^{-\frac{1}{2}} \cdot 2b = 0$$

$$\Rightarrow y + \frac{b}{\sqrt{a^2 + b^2 + 1}} = 0$$

$$\Rightarrow y = \frac{-b}{\sqrt{a^2 + b^2 + 1}} \quad —③$$

$$x^2 + y^2 = \frac{a^2}{a^2 + b^2 + 1} + \frac{b^2}{a^2 + b^2 + 1}$$

$$\dot{=} \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{+1}{a^2+b^2+1}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{a^2+b^2+1}}$$

$$\sqrt{a^2+b^2+1} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$② \Rightarrow \quad x = -a\sqrt{1-x^2-y^2} \Rightarrow a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$③ \Rightarrow \quad y = -b\sqrt{1-x^2-y^2} \Rightarrow b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Sub in ①

$$Z = \frac{-x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$x^2+y^2+z^2 = 1 \text{ is the singular}$$

solution.

To get the general integral

Put $b = f(a)$ in ①.

$$z = ax + f(a) y + \sqrt{1 + a^2 + (f(a))^2} \quad \rightarrow ④$$

Diff ④ p.w.r.to 'a',

$$0 = x + f'(a) y + \frac{1}{2} (1 + a^2 + (f(a))^2)^{-\frac{1}{2}} a$$

$$(2a + 2 f(a) \cdot f'(a))$$

$$0 = x + f'(a) y + \frac{a + f(a) f'(a)}{\sqrt{1 + a^2 + (f(a))^2}} \quad \rightarrow ⑤$$

Eliminate 'a' between ④ & ⑤ we get the general solution.

④ $z = px + qy - 2\sqrt{pq}$

Sol:

This eqn is of the form

$$z = px + qy + f(p, q)$$

The complete integral is

$$z = ax + by + f(a, b)$$

$$z = ax + by - 2\sqrt{ab} \quad \rightarrow ①$$

To find singular integral.

Diff p.w.r.to a & b in ①

$$\frac{\partial z}{\partial a} = 0$$

$$\Rightarrow x + 0 - 2 \frac{1}{2} (ab)^{-\frac{1}{2}} b = 0$$

$$\Rightarrow x = (ab)^{-\frac{1}{2}} \cdot b$$

$\frac{\partial z}{\partial b} = 0$

$\Rightarrow \quad y - 2 \cdot \frac{1}{2} (ab)^{-\frac{1}{2}} \cdot a = 0$

$\Rightarrow \quad y = (ab)^{-\frac{1}{2}} a$

$\Rightarrow \quad xy = (ab)^{\frac{1}{2}} \cdot (ab)^{-\frac{1}{2}} \cdot a \cdot b$

$= a^{-\frac{1}{2}} b^{-\frac{1}{2}} a^{-\frac{1}{2}} \cdot b^{-\frac{1}{2}} \cdot a \cdot b$

$= a^{-1} \cdot b^{-1} \cdot a \cdot b$

$\Rightarrow \quad xy = 1$

Type : 3   F(z, p, q) = 0.

This eqn is of the form $f(z, p, q) = 0 \quad \text{---(1)}$

Let $z = f(x + ay)$ be the solution of (1)

put $x + ay = u \quad \text{---(2)}$

Then $z = f(u) \quad \text{---(3)}$

Substitute $p = \dfrac{dz}{du}$ & $q = a \dfrac{dz}{du}$ Then

Integrating we get the solution.

---

1. Solve: $p(1+q) = qz$.

Sol:

Given $p(1+q) = qz \quad \text{---(1)}$

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$\dfrac{\partial u}{\partial x} = 1 \qquad \dfrac{\partial u}{\partial y} = a$

①

$$p = \frac{dz}{du} \quad , \quad q = a \frac{dz}{du}$$

$$① \Rightarrow \frac{dz}{du}\left(1 + a \frac{dz}{du}\right) = az \frac{dz}{du}$$

$$\therefore \quad 1 + a \frac{dz}{du} = az$$

$$a \frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az - 1}{a}$$

$$\frac{du}{dz} = \frac{a}{az - 1}$$

$$du = \frac{a}{az - 1} dz$$

$\underline{\text{Integrating}}$ on both sides, we get

$$u = \log(az - 1) + k.$$

$$x + ay = \log(az - 1).$$

② $z^2 = 1 + p^2 + q^2.$

Sol:

Given $z^2 = 1 + p^2 + q^2$ —①

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du} \qquad q = a \frac{dz}{du}$$

$$① \Rightarrow \quad z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (1+a^2) = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \left(\frac{z^2-1}{a^2+1}\right) \quad \rightarrow \text{(1)}$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z^2-1}{a^2+1}}$$

$$\Rightarrow \sqrt{a^2+1} \, dz = \sqrt{z^2-1} \, du$$

$$\Rightarrow \sqrt{a^2+1} \, \frac{dz}{\sqrt{z^2-1}} = du$$

Integrating,

$$\sqrt{a^2+1} \int \frac{dz}{\sqrt{z^2-1}} = \int du + b$$

$$\sqrt{a^2+1} \, \cosh^{-1}(z) = u + b$$

$$\sqrt{a^2+1} \, \cosh^{-1}(z) = x + ay + b$$

is the complete solution.

③ $p + q = z$.

Sol:

Given $p + q = z \rightarrow$ ①

Let $u = x + ay$

$$p = \frac{dz}{du} \qquad q = a\frac{dz}{du}$$

$$\frac{dz}{du} + a\frac{dz}{du} = z$$

$$\frac{dz}{du}(1+a) = z$$

$$\frac{dz}{du} = \frac{z}{1+a}$$

$$(1+a)\frac{dz}{z} = du + (a-1) \, a$$

Integrating,

$$(1+a)\int \frac{dz}{z} = \int du$$

$$(1+a)\log z = u + b$$

$$(1+a)\log z = x + ay + b. \text{ is the}$$

complete solution.

④ Solve: $p(1-q^2) = q(1-z)$

Sol:

Gn $p(1-q^2) = q(1-z)$ —①

Let $u = x + ay$

$$p = \frac{dz}{du} \quad ; \quad q = a\frac{dz}{du}$$

$$\frac{dz}{du}\left(1 - a^2\left(\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(1-z)$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a\frac{dz}{du}(1-z)$$

$$= a - az$$

$$a^2\left(\frac{dz}{du}\right)^2 = -a + az + 1$$

$$a\frac{dz}{du} = \sqrt{1 - a + az}$$

$$\frac{a}{\sqrt{1-a+az}}\, dz = du$$

$$a(1-a+az)^{-\frac{1}{2}} dz = du$$

Integrating we get

$$a \frac{(1-a+az)^{\frac{1}{2}}}{\frac{1}{2} \, a} = u+b$$

$$2(1-a+az)^{\frac{1}{2}} = x+ay+b \text{ is the}$$

complete solution.

Type: 3(a)

Equation containing (a, p, q) only

i) Let $q=a$

ii) Find p

iii) $dz = p\,dx + q\,dy$

iv) Integrate, we get the complete

Solution.

---

① Solve : $p = 2qx$

Sol: Gn $p = 2qx$

Let $q = a \Rightarrow p = 2ax$

$$dz = p\,dx + q\,dy$$

$$dz = 2ax\,dx + a\,dy$$

Integrate,

$$z = \frac{2ax^2}{2} + ay + b$$

$$z = ax^2 + ay + b \text{ is the complete}$$
solution

Diff p.w.r. to b,

$\qquad 0 = 1 \quad$ is absurd

$\qquad$ There is no singular solution.

② Solve: $q = 2px$

Sol:

$\qquad q = 2px$

$\qquad$ Let $q = a$ $\qquad\qquad a = 2px$

$\qquad q = 2pa$

$\qquad \Rightarrow p = \dfrac{q}{2a} \qquad \Rightarrow p = \dfrac{a}{2x}$

$\qquad dz = p\,dx + q\,dy$

$\qquad dz = \dfrac{a}{2x}\,dx + a\,dy$

Integrate,

$\qquad z = \dfrac{+a}{2x}(\log x) + ay + b$

$\qquad z = \dfrac{a}{2}\log x + ay + b$ is the

complete solution.

Diff p.w.r. to b,

$\qquad 0 = 1$ is absurd.

$\qquad$ There is no singular solution.

Type : 3 (c)

$\qquad$ Equation containing y, p, q only

i) Let $p = a$.

ii) find q

iii) $dz = p\,dx + q\,dy$

iv) Integrate we get the complete solution.

① Solve: $2yp^2 = q$

Sol:

$$2yp^2 = q$$

Let $p = a$

$$2ya^2 = q$$

$$q = 2ya^2$$

$$dz = pdx + qdy$$

$$dz = adx + 2ya^2 dy$$

Integrate,

$$z = ax + 2a^2 \frac{y^2}{2} + b$$

$$z = ax + a^2 y^2 + b \quad \text{is the}$$

Complete solution.

2. Solve: $q = py + p^2$

Sol:

$$q = py + p^2$$

Let $p = a$

$$q = ay + a^2$$

$$dz = pdx + qdy$$

$$= adx + (ay + a^2)dy$$

Integrate, $z = ax + \dfrac{ay^2}{2} + a^2 y + b$ is the

complete solution.

Diff p.w.r.to b. $0 = 1$ is absurd

There is no singular solution

Type:civ)

    Equation containing x, y, p, q.

i) Attach x & p in one side

ii) Attach y & q in other side

iii) Let it be equal to k

iv) Find p & q

v) $dz = p\,dx + q\,dy$

vi) Integrate we get the complete solution

① Solve: $p + q = x + y$

Sol: Gn

$$p - x = y - q = k$$

$$p - x = k, \quad y - q = k$$

$$p = x + k \qquad q = y - k$$

$$dz = p\,dx + q\,dy$$

$$dz = (x + k)\,dx + (y - k)\,dy$$

Integrating,

$$z = \frac{x^2}{2} + kx + \frac{y^2}{2} - ky + b \text{ is the}$$

complete solution

Diff p.w.r.to b, $0 = 1$ is absurd

    There is no singular solution

② Solve: $pq = xy$

Sol: $pq = xy$

$$\frac{p}{x} = \frac{q}{q} = k \text{ (say)}$$

$$\frac{p}{x} = k \quad, \quad \frac{q}{q} = k.$$

$$p = xk \qquad q = \frac{y}{k}$$

$$dz = p\,dx + q\,dy$$

$$= (xk)\,dx + \left(\frac{y}{k}\right)dy$$

Integrating we get

$$z = \frac{x^2}{2}k + \frac{y^2}{2k} + b \text{ is the}$$

complete solution.

Diff p.w.r. to b, $0 = 1$ is absurd

There is no singular integral.

③ Solve: $p^2 y (1+x^2) = qx^2$

sol: Gn $p^2 y (1+x^2) = q x^2$

$$\Rightarrow \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = k$$

$$\frac{p^2(1+x^2)}{x^2} = k \qquad \frac{q}{y} = k$$

$$1 + p^2 = \frac{kx^2}{1+x^2} \qquad q = yk.$$

$$p = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}\,dx + yk\,dy$$

Integrate.

$$Z = \sqrt{k} \int \frac{x}{\sqrt{1+x^2}} \, dx + k \int y \, dy + b$$

$$Z = \sqrt{k} \sqrt{1+x^2} + k \frac{y^2}{2} + b \text{ is the complete}$$

solution.

(4) $\sqrt{p} + \sqrt{q} = x + y$

Sol:

$$\sqrt{p} + \sqrt{q} = x + y$$

$$\sqrt{p} - x = y - \sqrt{q} = k$$

$$\sqrt{p} - x = k \qquad y - \sqrt{q} = k$$

$$\sqrt{p} = x + k \qquad \sqrt{q} = y - k$$

$$p = (x+k)^2 \qquad q = (y-k)^2$$

$$dz = p \, dx + q \, dy$$

$$dz = (x+k)^2 \, dx + (y-k)^2 \, dy$$

Integrate,

$$z = \frac{(x+k)^3}{3} + \frac{(y-k)^3}{3} + b \text{ is the}$$

complete solution.

Type - V

    form the equation of the type of

$$f(x^m p, y^n q, z) = 0$$

case: (i)

    $m \neq 1, n \neq 1$ then $X = x^{1-m}, Y = y^{1-n}$

case: (ii)

    $m = n = 1$ then put $X = \log x$

                                $Y = \log y$

Next we follow type (3)

⑥ Solve: $p^2 + x^2 y^2 q^2 = x^2 z^2$

Sol:

$$p^2 + x^2 y^2 q^2 = x^2 z^2$$

$\div x^2,$
$$\frac{p^2}{x^2} + y^2 q^2 = z^2$$

$$x^{-2} p^2 + y^2 q^2 = z^2$$

$$(x^{-1} p)^2 + (yq)^2 = z^2 \longrightarrow ①$$

This is of the form $f(x^m p, y^n q, z) = 0$

Here $m = -1, \quad n = 1$

put $X = x^{1-m}$ $Y = y^n \log y$

$X = x^{1+1}$ $Y = y^1 \log y$

$X = x^2$ $Y = \log y$

$$\frac{\partial X}{\partial x} = 2x \qquad Q = \frac{\partial z}{\partial Y}$$

$$p = \frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial X}{\partial x} \qquad q = Q \frac{1}{y}$$

$$p = 2x \cdot P \qquad yq = Q$$

$$\frac{p}{2x} = P$$

$$x^{-1} p = 2P$$

① $\Rightarrow (2P)^2 + Q^2 = z^2 \longrightarrow ②$

This is of the form $f(P, Q, z) = 0$

We use Type (3)

Let $u = x + ay$

$\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial u}{\partial y} = a$

$P = \dfrac{\partial z}{\partial x} = \dfrac{dz}{du} \cdot \dfrac{\partial u}{\partial x}$

$P = \dfrac{dz}{du}$

$Q = \dfrac{\partial z}{\partial y} = \dfrac{dz}{du} \cdot \dfrac{\partial u}{\partial y}$

$Q = a \dfrac{dz}{du}$

(2) $\Rightarrow$ $\left(2\dfrac{dz}{du}\right)^2 + \left(a\dfrac{dz}{du}\right)^2 = z^2$

$(4 + a^2)\left(\dfrac{dz}{du}\right)^2 = z^2$

$\left(\dfrac{dz}{du}\right)^2 = \dfrac{z^2}{4+a^2}$

$\dfrac{dz}{du} = \dfrac{z}{\sqrt{4+a^2}}$

$\dfrac{dz}{z} = \dfrac{1}{\sqrt{4+a^2}} du$

$\displaystyle\int \dfrac{dz}{z} = \int \dfrac{1}{\sqrt{4+a^2}} du$

$\log z = \dfrac{1}{\sqrt{4+a^2}} u + b$

$\log z = \dfrac{1}{\sqrt{4+a^2}} (x + ay) + b$

$\log z = \dfrac{1}{\sqrt{4+a^2}} (x^2 + a \log y) + b$

2. Solve: $x^2 p^2 + y^2 q^2 = z^2$

   Sol:
$$x^2 p^2 + y^2 q^2 = z^2$$
$$(xp)^2 + (yq)^2 = z^2 \longrightarrow \textcircled{1}$$

This eqn is of the form $f(x^m p, y^n q, z) = 0$

Here $m = 1$, $n = 1$.

Put $x = \log x$ $y = \log y$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

$$P = \frac{\partial z}{\partial X} \qquad Q = \frac{\partial z}{\partial Y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$$p = P \cdot \frac{1}{x} \qquad q = Q \cdot \frac{1}{y}$$
$$xp = P \qquad yq = Q$$

Sub in eqn ①, we get
$$P^2 + Q^2 = z^2 \longrightarrow \textcircled{2}$$

This eqn is of the form $f(P, Q, z) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = a$$

$$P = \frac{\partial u}{\partial z} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \qquad Q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$P = \frac{dz}{du} \qquad Q = a \frac{dz}{du}$$

Sub in ② we get

$$\left(\frac{dz}{du}\right)^2 + \left(a\,\frac{dz}{du}\right)^2 = z^2 \longrightarrow ③$$

$$\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{1+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

$$\int \frac{dz}{z} = \int \frac{1}{\sqrt{1+a^2}}\,du$$

$$\log z = \frac{1}{\sqrt{1+a^2}}\,u + c$$

$$\log z = \frac{1}{\sqrt{1+a^2}}(x+ay) + c$$

$$\log z = \frac{1}{\sqrt{1+a^2}}(\log x + a\log y) + c$$

which is the complete solution.

Type: 6

Eqn of the type $f(z^m p, z^m q) = 0 \longrightarrow ①$

& $f_1(x, z^m p) = f_2(y, z^m q) \longrightarrow ②$

case:(i) if $m \neq -1$ put $z = z^{m+1} \Rightarrow$ Type ① $\Rightarrow$ Type ④

case:(ii) if $m = -1$ then put $x = \log z$

$\Rightarrow$ Type ① $\Rightarrow$ Type ④

① Solve $z^2(p^2+q^2) = x^2+y^2$

Sol.

Given $z^2(p^2+q^2) = x^2+y^2$

$\qquad (zp)^2 + (zq)^2 = x^2+y^2 \longrightarrow ①$

This eqn is of the form

$\qquad f_1(x, z^m p) = f_2(y, z^m q)$

Here $M \neq -1$,

$\qquad$ Put $z = z^{m+1}$

$\qquad \Rightarrow z = z^{1+1} = z^2.$

$\qquad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial z} \cdot \dfrac{\partial z}{\partial x}$

$\qquad p = 2zp$

$\qquad \dfrac{P}{2} = zp$

Similarly, $\dfrac{Q}{2} = zq$

Sub in eqn ①, we get

$\qquad \left(\dfrac{P}{2}\right)^2 + \left(\dfrac{Q}{2}\right)^2 = x^2+y^2$

$\qquad P^2 + Q^2 = 4(x^2+y^2)$

$\qquad P^2 - 4x^2 = -Q^2 + 4y^2.$

This eqn is of the form $f_1(x, P) = f_2(y, Q)$

$\quad \therefore\ P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2$

$\quad P^2 = 4a^2 + 4x^2 \qquad\qquad Q^2 = -4a^2 + 4y^2$

$\quad P = 2\sqrt{a^2+x^2} \qquad\qquad Q = 2\sqrt{y^2-a^2}$

$$dz = P dx + Q dy$$

$$dz = 2\sqrt{a^2 + x^2}\, dx + 2\sqrt{y^2 - a^2}\, dy$$

$$\int dz = 2 \int \sqrt{a^2 + x^2}\, dx + 2 \int \sqrt{y^2 - a^2}\, dy$$

$$z = 2\left[\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} \right.$$
$$\left. - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) \right] + b$$

$$z^2 = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2 - a^2}$$
$$- a^2 \cosh^{-1}\left(\frac{y}{a}\right) + b$$

$$= x\sqrt{x^2 + a^2} + y\sqrt{y^2 - a^2} + a^2 \left[\sinh^{-1}\left(\frac{x}{a}\right) - \cosh^{-1}\left(\frac{y}{a}\right) \right] + b$$

---

2. Solve : $p^2 + q^2 = z^2(x^2 + y^2)$

Sol:

$$p^2 + q^2 = z^2(x^2 + y^2) \longrightarrow ①$$

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2$$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^n q)$$

Here $m = -1$.

put $z = \log z$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$P = \frac{1}{z} \cdot p$$

Similarly, $Q = \frac{1}{z} \cdot q$

Sub in eqn ① , we get

$$P^2 - x^2 = y^2 - Q^2$$

This eqn is of the form

$$f_1(x, P) = f_2(y, Q) \quad \text{Type } \textcircled{4}$$

$$P^2 - x^2 = y^2 - Q^2 = a^2$$

$$P^2 - x^2 = a^2 \qquad y^2 - Q^2 = a^2$$

$$P^2 = x^2 + a^2 \qquad\qquad Q^2 = y^2 - a^2$$

$$P = \sqrt{x^2 + a^2} \qquad\qquad Q = \sqrt{y^2 - a^2}$$

$$dz = Pdx + Qdy$$

$$dz = \sqrt{x^2 + a^2}\, dx + \sqrt{y^2 - a^2}\, dy$$

$$\int dz = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{y^2 - a^2}\, dy$$

$$z = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 + x^2}$$

$$+ \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b.$$

$$(\because z = \log z$$

$$\log z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{a^2 + x^2}$$

$$+ \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b.$$

⑪ Lagrange's Linear Equation:

method of Grouping:

In the auxilary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{if the variables can}$$

be separated in any pair of equations, then we get a solution of the form

$$u(x,y) = a \quad \& \quad v(x,y) = b.$$

① Solve: $px + qy = z$

Sol: Given $px + qy = z$

$$P = x, \quad Q = y, \quad R = z$$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$x = (y \cdot c_1)$$

$$\Rightarrow \frac{x}{y} = c_1 \quad (iv) \ u = \frac{x}{y}$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$$y = z c_2$$

$$\Rightarrow \frac{y}{z} = c_2 \quad (iv) \ v = \frac{y}{z}$$

The solution of the given P.de is

$$\varphi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

② Write the general integral of

$$pyz + qzx = xy.$$

Sol: The given eqn is of the form

$$Pp + Qq = R.$$

$$P = yz, \quad Q = zx, \quad R = xy$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$

$$\Rightarrow \int x\,dx = \int y\,dy$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\Rightarrow c_1 = x^2 - y^2$$

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow c_2 = y^2 - z^2$$

$$\therefore \varphi(x^2 - y^2, \; y^2 - z^2) = 0.$$

③ Solve: $x(y-z)p + y(z-x)q = z(x-y)$

Sol: The given eqn is of the form

$$Pp + Qq = R$$

$$P = x(y-z) \quad Q = y(z-x) \quad R = z(x-y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} = Each\ ratio$$

$$dx+dy+dz = 0$$

Integrating, $x+y+z+c_1$

$$Each\ ratio = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $\log x + \log y + \log z = \log c_2$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\therefore \quad \varphi(x+y+z, \ xyz) = 0.$$

(A) $(x^2-yz)p + (y^2-zx)q = z^2-xy.$

Sol:
     This eqn is of the form $Pp+Qq = R$

$$P = x^2-yz \qquad Q = y^2-zx \qquad R = z^2-xy$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xyz+y^3-xyx+z^3-xyz}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz+zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz+zx}$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)(dx+dy+dz)$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + C$$

$$x^2 + y^2 + z^2 = (x+y+z)^2 + C$$

$$x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + C$$

$$-2(xy + yz + zx) = C$$

$$xy + yz + zx = u \ (\text{constant})$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\frac{d(x-y)}{(x^2-y^2) + z(x-y)} = \frac{d(y-z)}{y^2-z^2 + x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(y+z+x)}$$

Integrating on both sides, we get ⑫

$$\log (x-y) = \log (y-z) + \log c$$

$$\log (x-y) - \log (y-z) = \log c$$

$$\log \left(\frac{x-y}{y-z} \right) = \log c$$

$$\frac{x-y}{y-z} = c.$$

The general solution is

$$\phi \left(xy + yz + zx, \frac{x-y}{y-z} \right) = 0.$$

Method of Multipliers:

choose any three multipliers l, m, n which may be constants or functions of x, y, z, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

It is possible to choose l, m, n such that

$$lP + mQ + nR = 0 \text{ then } l\,dx + m\,dy + n\,dz = 0$$

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on integration we get a solution

$u = a.$
    The multipliers l, m, n are called Lagrangian multipliers.

(1) Solve: $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

Sol:

Given $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

This is of the form

$$Pp + Qq = R.$$

where $P = x(y^2-z^2) \quad Q = y(z^2-x^2) \quad R = z(x^2-y^2)$

The subsidiary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Each ratio $= \dfrac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)}$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2y^2 - x^2z^2 + y^2z^2 - y^2x^2 + z^2x^2 - z^2y^2}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c.$$

Each ratio $= \dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_1$$

$$xyz = c_1$$

$$\therefore \phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, \, xyz\right) = 0.$$

2. Solve: $(m^2 - ny)p + (nx - lz)q = ly - mx$

Sol:

Given $(m^2 - ny)p + (nx - lz)q = ly - mx$

This eqn is of the form $Pp + Qq = R$

where $P = m^2 - ny$, $Q = nx - lz$, $R = ly - mx$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{m^2 - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Each ratio $= \dfrac{dx + dy + dz}{m^2 - ny + nx - lz + ly - mx}$

$$= \frac{x\,dx + y\,dy + z\,dz}{xmz - xny + nyx - ylz + lyz - mxz}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0.$$

Integrating.

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c.$$

Each ratio $= \dfrac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + lny - mnx}$

$$\Rightarrow ldx + mdy + ndz = 0$$

Integrating.

$$lx + my + nz = C_1$$

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, \; lx + my + nz\right) = 0.$$

⑧ Solve: $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

Sol:

This eqn is of the form $Pp + Qq = R$

$$P = 3z - 4y, \quad Q = 4x - 2z, \quad R = 2y - 3x$$

$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

Each ratio $= \dfrac{xdx + ydy + zdz}{3xz - 4yx + 4xy - 2yz + 2yz - 3xz}$

$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Each ratio $= \dfrac{2dx + 3dy + 4dz}{6z - 8y + 12x - 6z + 8y - 12x}$

$$\Rightarrow 2dx + 3dy + 4dz = 0.$$

$2x + 3y + 4z = c_2$.

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, \ 2x + 3y + 4z\right) = 0.$$

(4) Solve: $(y - xz)p + (yz - x)q = (x+y)(x-y)$

Sol:

This eqn is of the form $Pp + Qq = R$

$P = y - xz$, $Q = yz - x$, $R = (x+y)(x-y)$

$$\frac{dx}{y - xz} = \frac{dy}{yz - x} = \frac{dz}{(x+y)(x-y)} = \frac{dz}{x^2 - y^2}$$

Each ratio $= \dfrac{xdx + ydy + zdz}{xy - x^2z + y^2z - xy + x^2z - zy^2}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C.$$

$$\Rightarrow u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}.$$

Each ratio $= \dfrac{ydx + xdy + dz}{y^2 - xyz + xyz - x^2 + x^2 - y^2}$

$y\,dx + x\,dy + dz = 0$.

Integrating,

$yx + xy + z = \cdot$