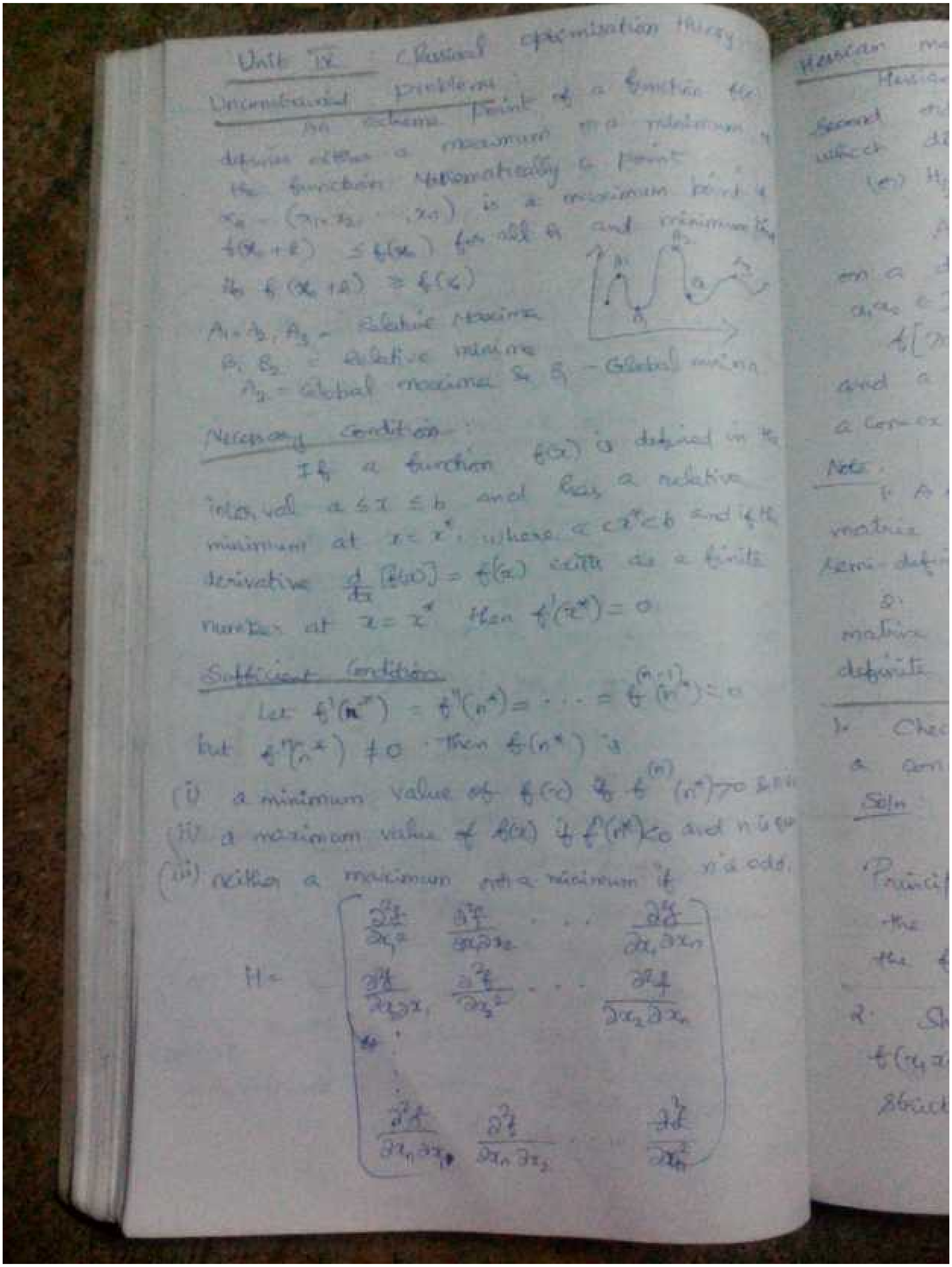

Unit IV : Classical optimization theory

Unconstrained problems

An extreme point of a function $f(x)$ defines either a maximum or a minimum of the function. Mathematically a point $x_0 = (x_1, x_2, \ldots, x_n)$ is a maximum point if $f(x_0 + h) \le f(x_0)$ for all h and minimum if $f(x_0 + h) \ge f(x_0)$

A_1, A_2, A_3 - Relative maxima

B_1, B_2 - Relative minima

A_2 - Global maxima & B_1 - Global minima

Necessary condition:

If a function $f(x)$ is defined in the interval $a \le x \le b$ and has a relative minimum at $x = x^*$, where $a < x^* < b$ and if the derivative $\frac{d}{dx}[f(x)] = f'(x)$ exists as a finite number at $x = x^*$ then $f'(x^*) = 0$

Sufficient Condition

Let $f'(x^*) = f''(x^*) = \cdots = f^{(n-1)}(x^*) = 0$ but $f^{n}(x^*) \ne 0$. Then $f(x^*)$ is

(i) a minimum value of $f(x)$ if $f^{(n)}(x^*) > 0$ & n is

(ii) a maximum value of $f(x)$ if $f^{(n)}(x^*) < 0$ and n is even

(iii) neither a maximum nor a minimum if n is odd.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

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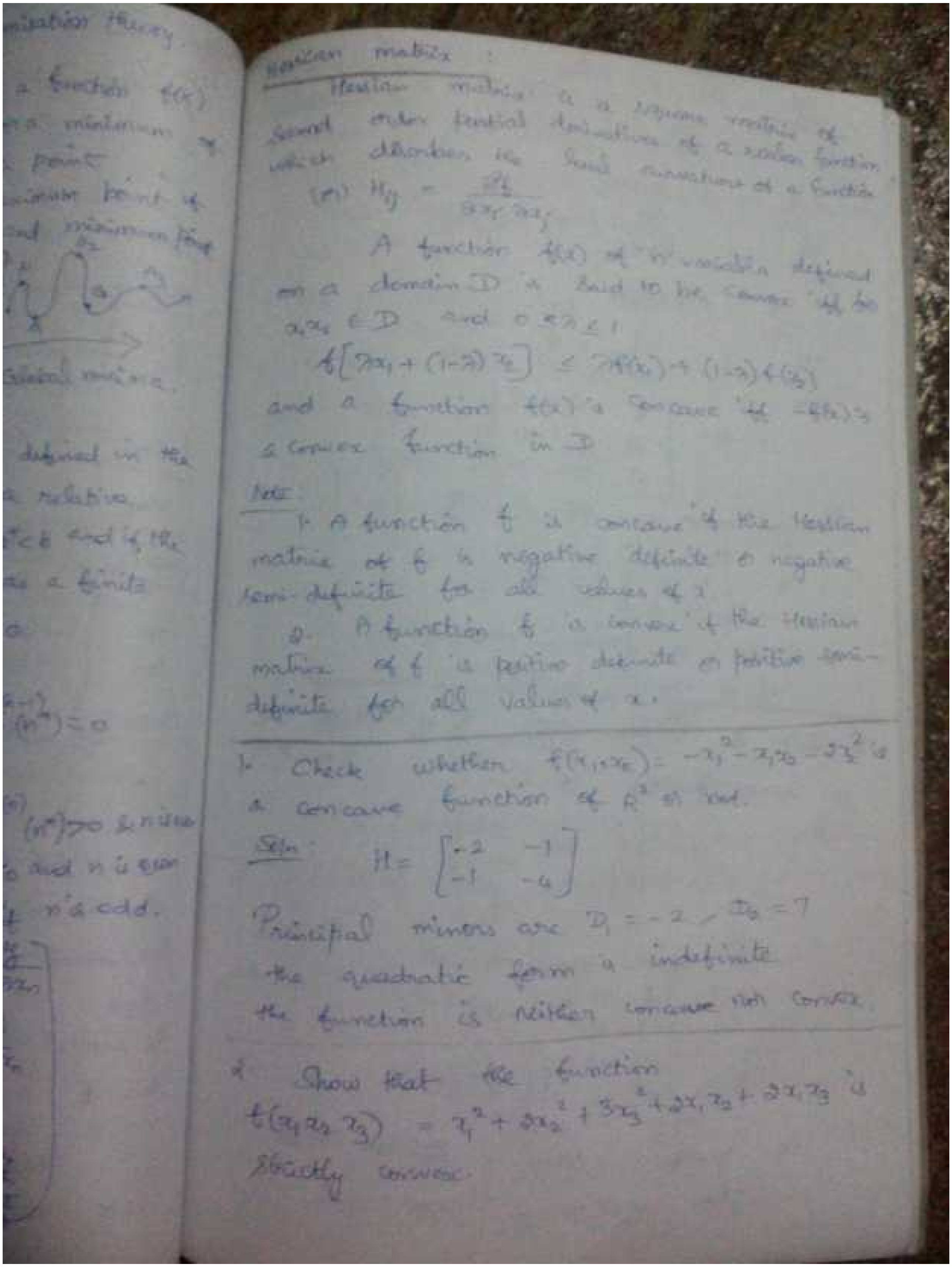


Hessian matrix :

Hessian matrix is a square matrix of second order partial derivatives of a scalar function which describes the local curvature of a function

(or) $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

A function f(x) of 'n' variables defined on a domain D is said to be convex iff for $x_1, x_2 \in D$ and $0 \le \lambda \le 1$

$$f[\lambda x_1 + (1-\lambda) x_2] \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

and a function f(x) is concave iff $-f(x)$ is a convex function in D

Note:

1. A function f is concave if the Hessian matrix of f is negative definite or negative semi-definite for all values of x

2. A function f is convex if the Hessian matrix of f is positive definite or positive semi-definite for all values of x.

---

1. Check whether $f(x_1, x_2) = -x_1^2 - x_1 x_2 - 2x_2^2$ is a concave function of R^2 or not.

Soln: $H = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$

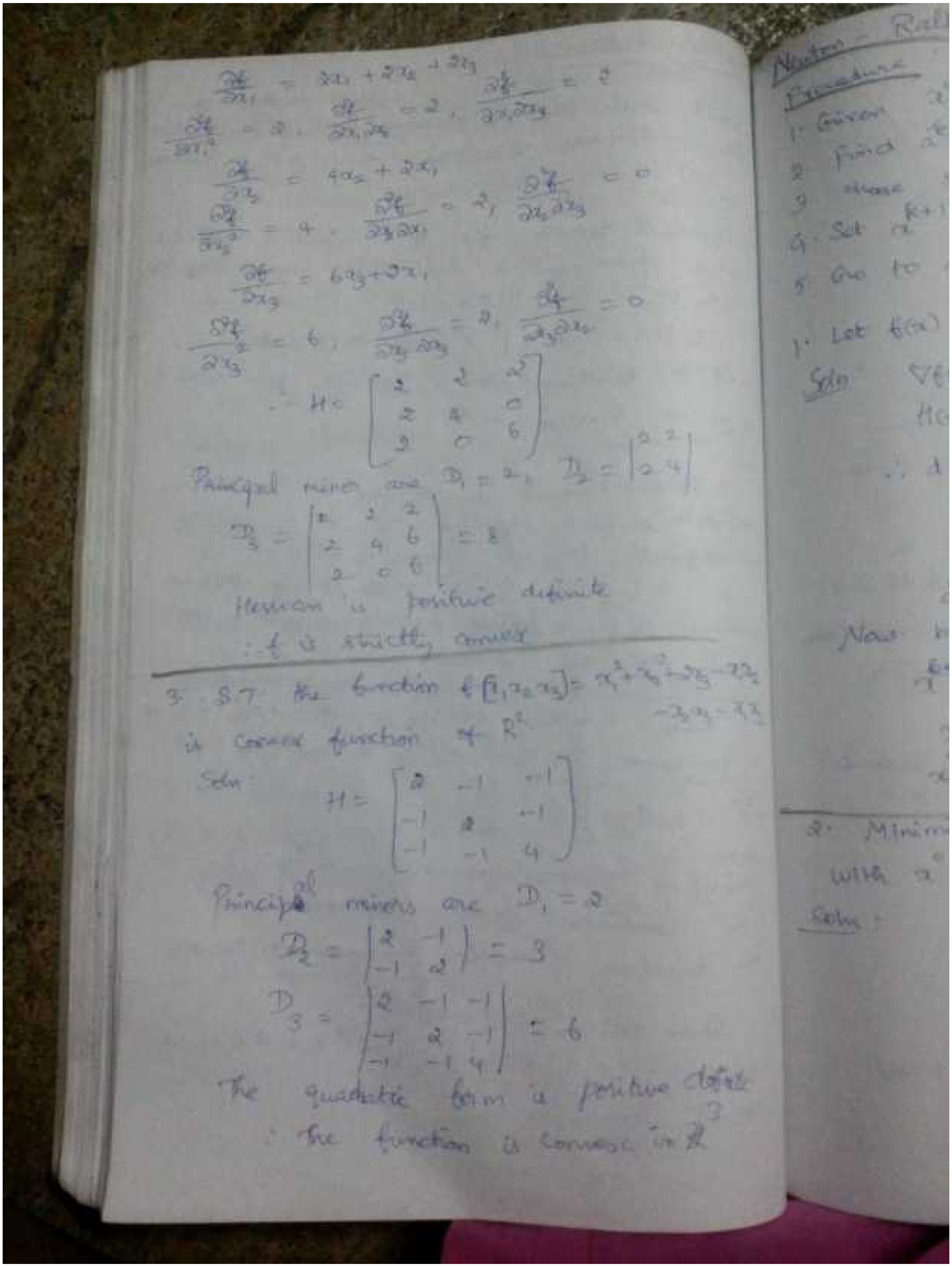
Principal minors are $D_1 = -2$, $D_2 = 7$

the quadratic form is indefinite

the function is neither concave nor convex.

---

2. Show that the function $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3$ is strictly convex.



$\frac{\partial f}{\partial x_1} = 2x_1 + 2x_2 + 2x_3$

$\frac{\partial^2 f}{\partial x_1^2} = 2, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2, \quad \frac{\partial^2 f}{\partial x_1 \partial x_3} = 2$

$\frac{\partial f}{\partial x_2} = 4x_2 + 2x_1$

$\frac{\partial^2 f}{\partial x_2^2} = 4, \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 2, \quad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 0$

$\frac{\partial f}{\partial x_3} = 6x_3 + 2x_1$

$\frac{\partial^2 f}{\partial x_3^2} = 6, \quad \frac{\partial^2 f}{\partial x_3 \partial x_1} = 2, \quad \frac{\partial^2 f}{\partial x_3 \partial x_2} = 0$

$$\therefore H = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

Principal minors are $D_1 = 2, \quad D_2 = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix}$

$$D_3 = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & 0 & 6 \end{vmatrix} = 8$$

Hessian is positive definite

$\therefore f$ is strictly convex

---

3. S.T. the function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 - x_1x_2 - x_2x_3 - x_1x_3$ is convex function of R^3

Soln:

$$H = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

Principal minors are $D_1 = 2$

$$D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$D_3 = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 4 \end{vmatrix} = 6$$

The quadratic form is positive definite

$\therefore$ the function is convex in R^3

Newton - Ral

Procedure

1. Given x

2. Find x^k

3. Choose

4. Set x^{k+1}

5. Go to

1. Let $f(x)$

Soln ∇f

$H($

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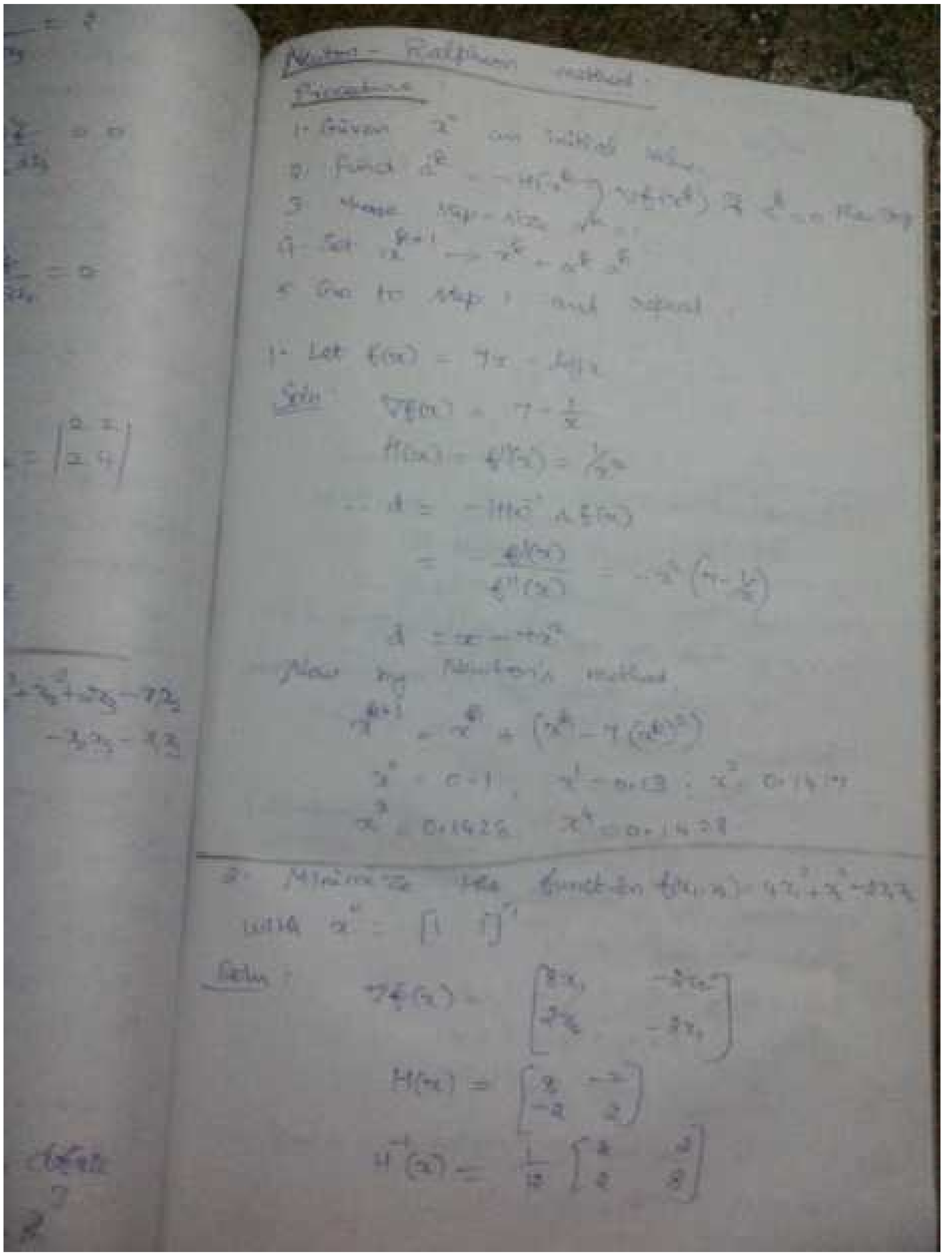
x^{k+}

2. Minim

with x^0

Soln:



Newton - Raphson method

Procedure

1. Given x^0 an initial value
2. Find $d^k = -H(x^k)^{-1}\nabla f(x^k)$, if $d^k = 0$ then stop
3. [illegible] step-size α^k
4. Set $x^{k+1} \to x^k + \alpha^k d^k$
5. Go to step 1 and repeat

1. Let $f(x) = 7x - \log x$

Soln: $\nabla f(x) = 7 - \frac{1}{x}$

$H(x) = f''(x) = \frac{1}{x^2}$

$\therefore d = -H(x)^{-1} \nabla f(x)$

$= -\frac{f'(x)}{f''(x)} = -x^2\left(7 - \frac{1}{x}\right)$

$d = x - 7x^2$

Now by Newton's method

$x^{k+1} = x^k + (x^k - 7(x^k)^2)$

$x^0 = 0.1$, $x^1 = 0.13$; $x^2 = 0.1417$

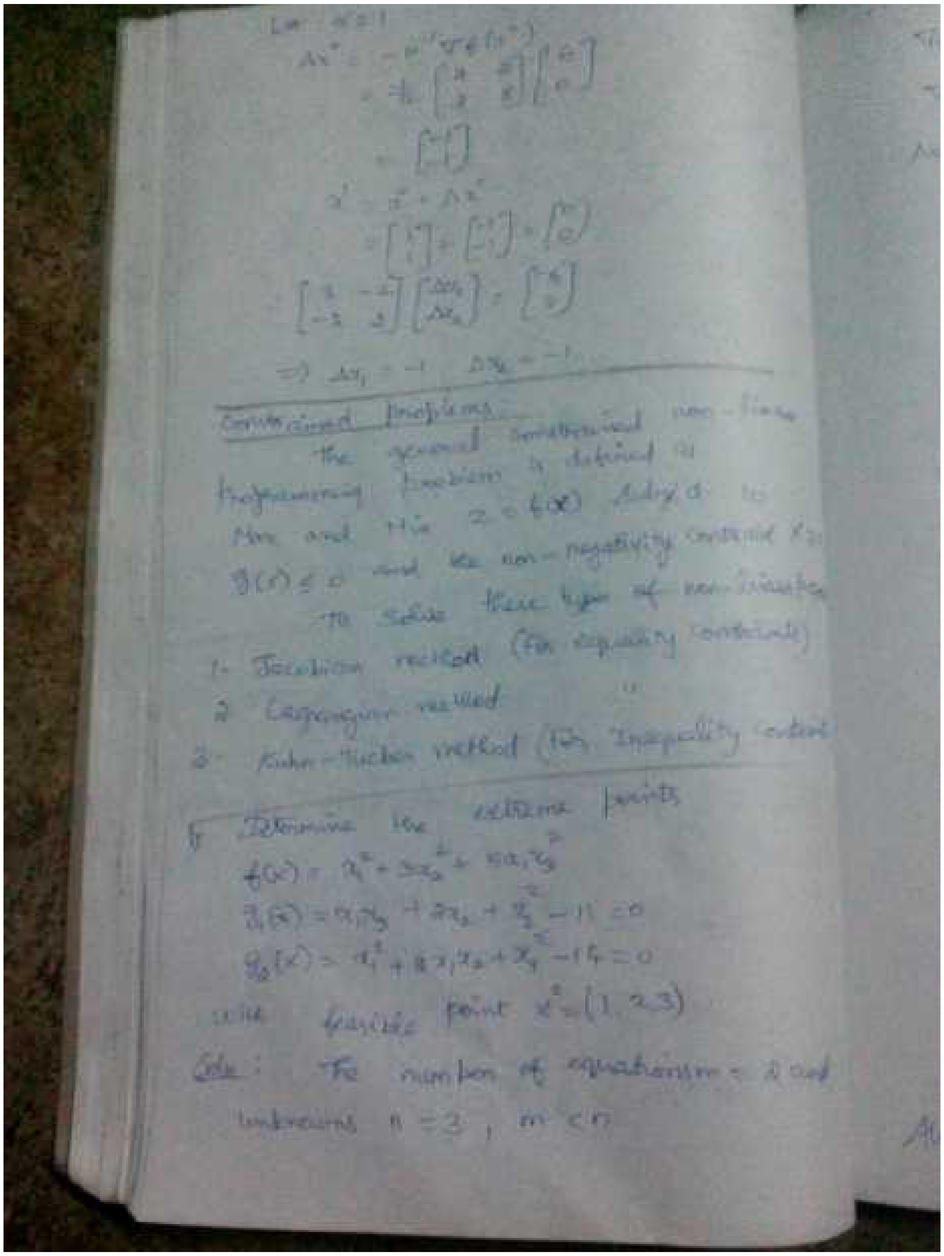
$x^3 = 0.1428$ $x^4 = 0.1428$

2. Minimize the function $f(x_1,x_2) = 4x_1^2 + x_2^2 - 2x_1x_2$ with $x^0 = [1 \quad 1]^T$

Soln: $\nabla f(x) = \begin{bmatrix} 8x_1 & -2x_2 \\ 2x_2 & -2x_1 \end{bmatrix}$

$H(x) = \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix}$

$H^{-1}(x) = \frac{1}{12}\begin{bmatrix} 2 & 2 \\ 2 & 8 \end{bmatrix}$



$\Delta x^0 = -J^{-1} \nabla f(x^0)$

[illegible]

$x^1 = x^0 + \Delta x^0$

[illegible]

$\Rightarrow \Delta x_1 = -1 \quad \Delta x_2 = -1$

Constrained problems

The general constrained non-linear programming problem is defined as Max and Min $z = f(x)$ subject to $g(x) \leq 0$ and the non-negativity constraint $x \geq 0$.

To solve these type of non-linear

1. Jacobian method (for equality constraints)
2. Lagrangian method "
3. Kuhn-Tucker method (for inequality constraints)

Determine the extreme points

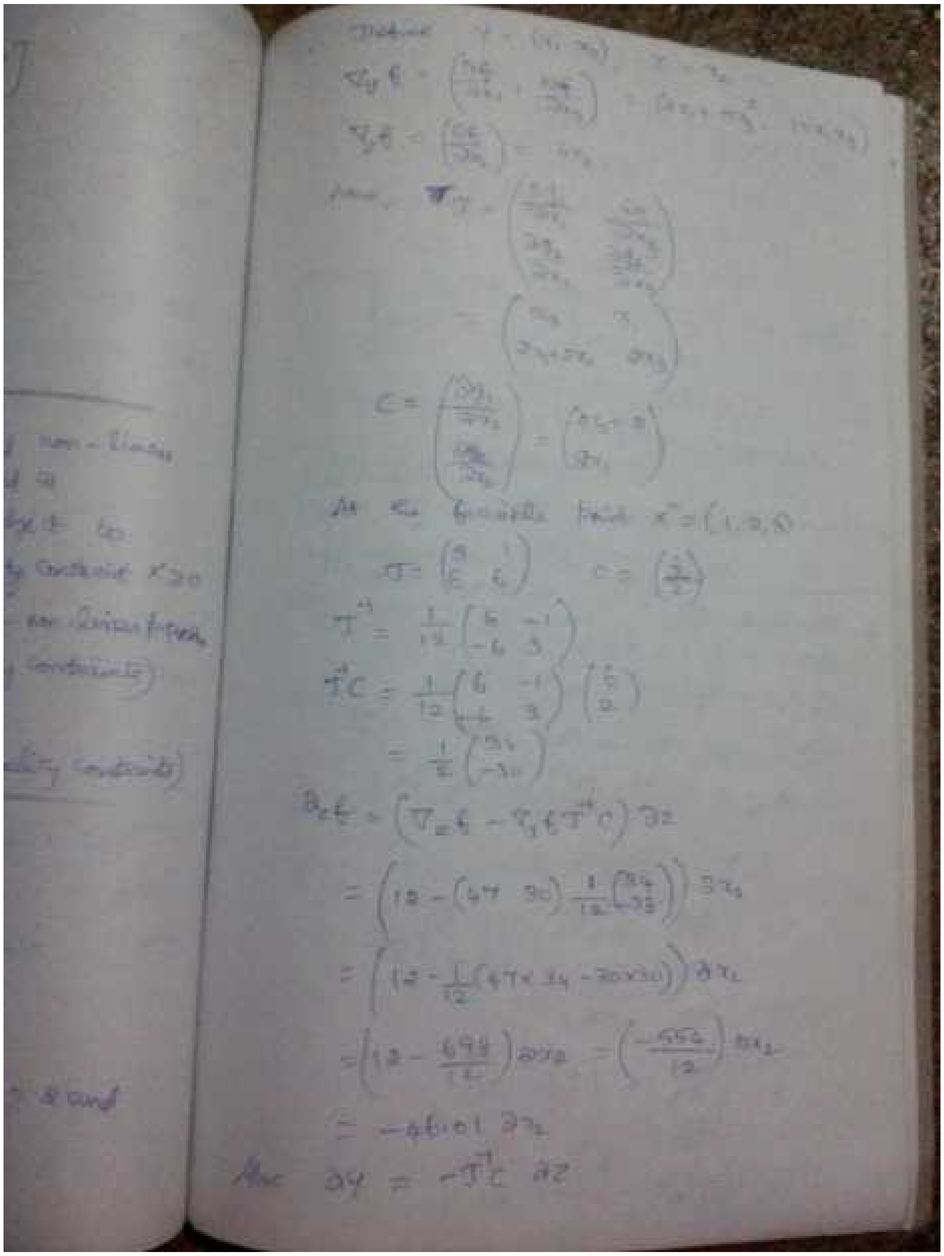
$f(x) = x_1^2 + 3x_2^2 + 5x_1x_3^2$

$g_1(x) = x_1x_3 + 2x_2 + x_2^2 - 11 = 0$

$g_2(x) = x_1^2 + 2x_1x_2 + x_3^2 - 14 = 0$

with feasible point $x^0 = (1, 2, 3)$

Soln: The number of equations $m = 2$ and unknowns $n = 3$, $m < n$



non-linear
subject to
constraint $x \geq 0$
non-linear fn
constraints)

ality constraints)

& and

Define $y = (x_1, x_2)$, $z = x_3$

$\nabla_y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) =$ [illegible]

$\nabla_z f = \left(\frac{\partial f}{\partial x_3}\right) =$ [illegible]

Now, $\nabla_y = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{pmatrix}$

$= \begin{pmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{pmatrix}$

$C = \begin{pmatrix} \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} [illegible] \\ 2x_1 \end{pmatrix}$

At the feasible point $x^\circ = (1, 2, 5)$

$J = \begin{pmatrix} 3 & 1 \\ 6 & 6 \end{pmatrix}$ $C = \begin{pmatrix} [illegible] \\ 2 \end{pmatrix}$

$J^{-1} = \frac{1}{12}\begin{pmatrix} 6 & -1 \\ -6 & 3 \end{pmatrix}$

$J^{-1}C = \frac{1}{12}\begin{pmatrix} 6 & -1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$= \frac{1}{12}\begin{pmatrix} 34 \\ -30 \end{pmatrix}$

$\partial_c f = \left(\nabla_z f - \nabla_y f J^{-1} C\right) \partial z$

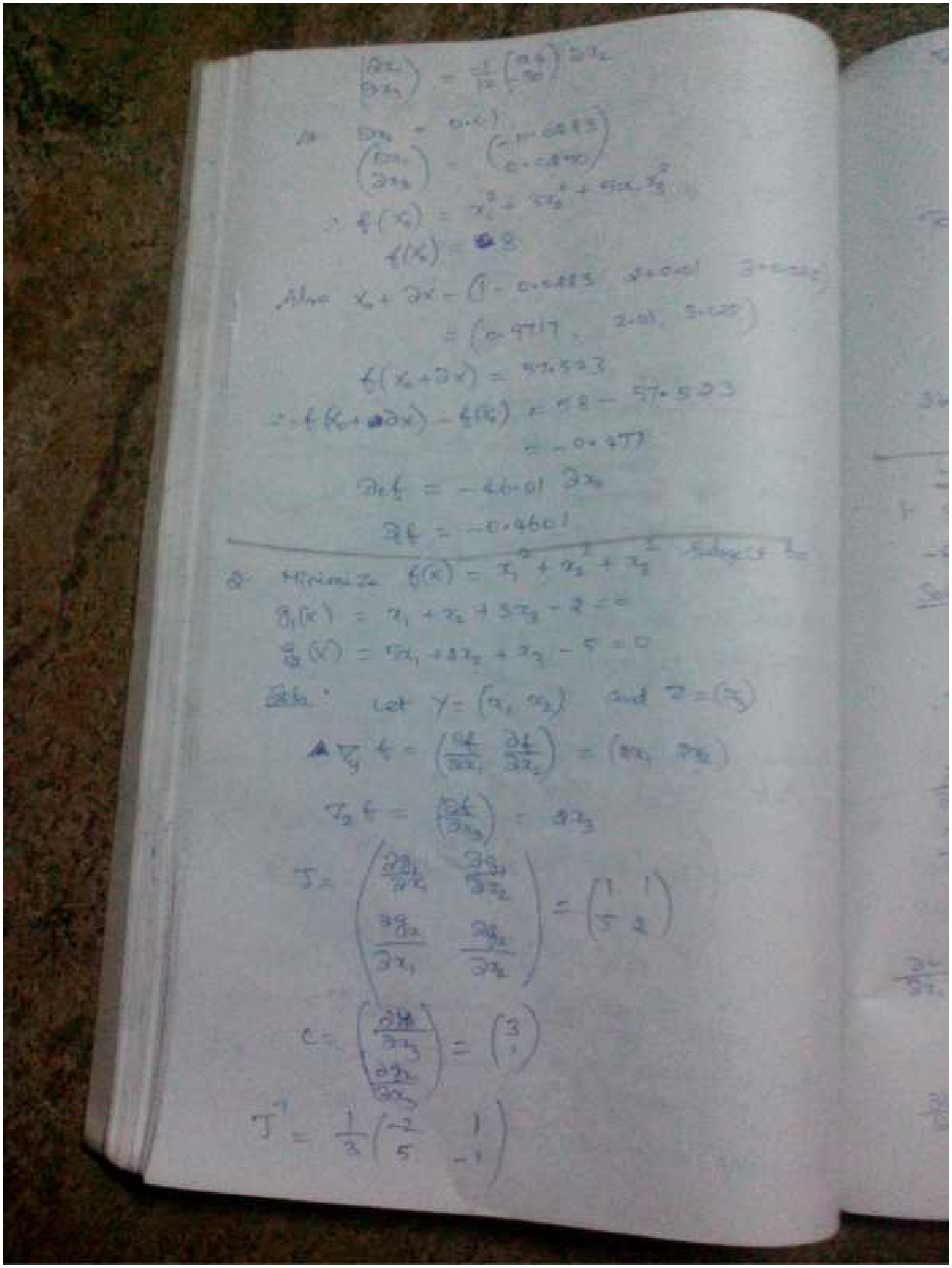
$= \left(12 - (47 \quad 30)\frac{1}{12}\begin{pmatrix} 34 \\ -30 \end{pmatrix}\right) \partial x_3$

$= \left(12 - \frac{1}{12}(47 \times 34 - 30 \times 30)\right) \partial x_3$

$= \left(12 - \frac{698}{12}\right) \partial x_3 = \left(-\frac{552}{12}\right) \partial x_3$

$= -46.01 \; \partial x_3$

Also $\partial y = -J^{-1}C \; \partial z$



$$\begin{pmatrix}\partial x_1\\ \partial x_3\end{pmatrix} = \frac{1}{12}\begin{pmatrix}34\\ -30\end{pmatrix}\partial x_2$$

$\partial x_2 = 0.01$

$$\begin{pmatrix}\partial x_1\\ \partial x_3\end{pmatrix} = \begin{pmatrix}-0.0283\\ 0.0250\end{pmatrix}$$

$\therefore f(X_0) = x_1^2 + 5x_2^2 + 5x_1x_3^2$

$f(X_0) = 58$

Also $X_0 + \partial X = (1-0.0283,\ 2+0.01,\ 3+0.025)$

$= (0.9717,\ 2.01,\ 3.025)$

$f(X_0+\partial X) = 57.523$

$\therefore f(X_0+\partial X) - f(X_0) = 58 - 57.523$

$= -0.477$

$\partial_c f = -46.01\,\partial x_2$

$\partial f = -0.4601$

Q. Minimize $f(X) = x_1^2 + x_2^2 + x_3^2$ subject to

$g_1(X) = x_1 + x_2 + 3x_3 - 2 = 0$

$g_2(X) = 5x_1 + 2x_2 + x_3 - 5 = 0$

Sol: Let $Y = (x_1, x_2)$ and $Z = (x_3)$

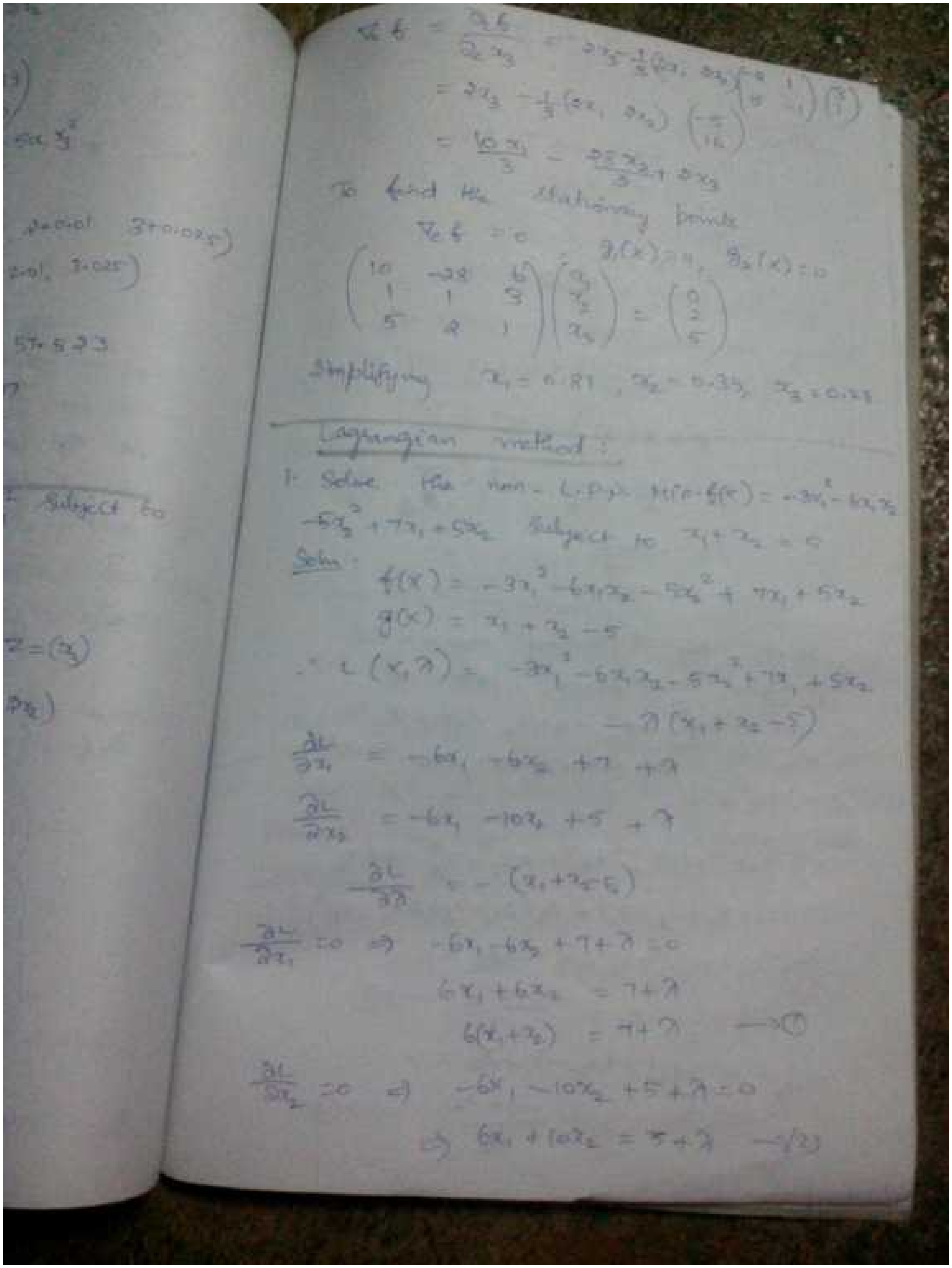
$\nabla_Y f = \left(\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2}\right) = (2x_1,\ 2x_2)$

$\nabla_Z f = \left(\frac{\partial f}{\partial x_3}\right) = 2x_3$

$$J = \begin{pmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 5 & 2\end{pmatrix}$$

$$C = \begin{pmatrix}\frac{\partial g_1}{\partial x_3}\\ \frac{\partial g_2}{\partial x_3}\end{pmatrix} = \begin{pmatrix}3\\ 1\end{pmatrix}$$

$$J^{-1} = \frac{1}{3}\begin{pmatrix}-2 & 1\\ 5 & -1\end{pmatrix}$$



$$\nabla_c f = \frac{\partial f}{\partial_c x_3} = 2x_3 - \frac{1}{3}(2x_1,\ 2x_2)\begin{pmatrix}-8 & 1\\ 5 & -1\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}$$

$$= 2x_3 - \frac{1}{3}(2x_1,\ 2x_2)\begin{pmatrix}-5\\16\end{pmatrix}$$

$$= \frac{10x_1}{3} - \frac{28x_2}{3} + 2x_3$$

To find the stationary points

$\nabla_c f = 0$, $g_1(x) = 0$, $g_2(x) = 0$

$$\begin{pmatrix}10 & -28 & 6\\ 1 & 1 & 3\\ 5 & 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\2\\5\end{pmatrix}$$

Simplifying $x_1 = 0.81$, $x_2 = 0.35$, $x_3 = 0.28$

Lagrangian method :

1. Solve the non-L.P.P. Min $f(x) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$ Subject to $x_1 + x_2 = 5$

Soln:

$f(x) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2$

$g(x) = x_1 + x_2 - 5$

$\therefore L(x, \lambda) = -3x_1^2 - 6x_1x_2 - 5x_2^2 + 7x_1 + 5x_2 - \lambda(x_1 + x_2 - 5)$

$\frac{\partial L}{\partial x_1} = -6x_1 - 6x_2 + 7 + \lambda$

$\frac{\partial L}{\partial x_2} = -6x_1 - 10x_2 + 5 + \lambda$

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 5)$

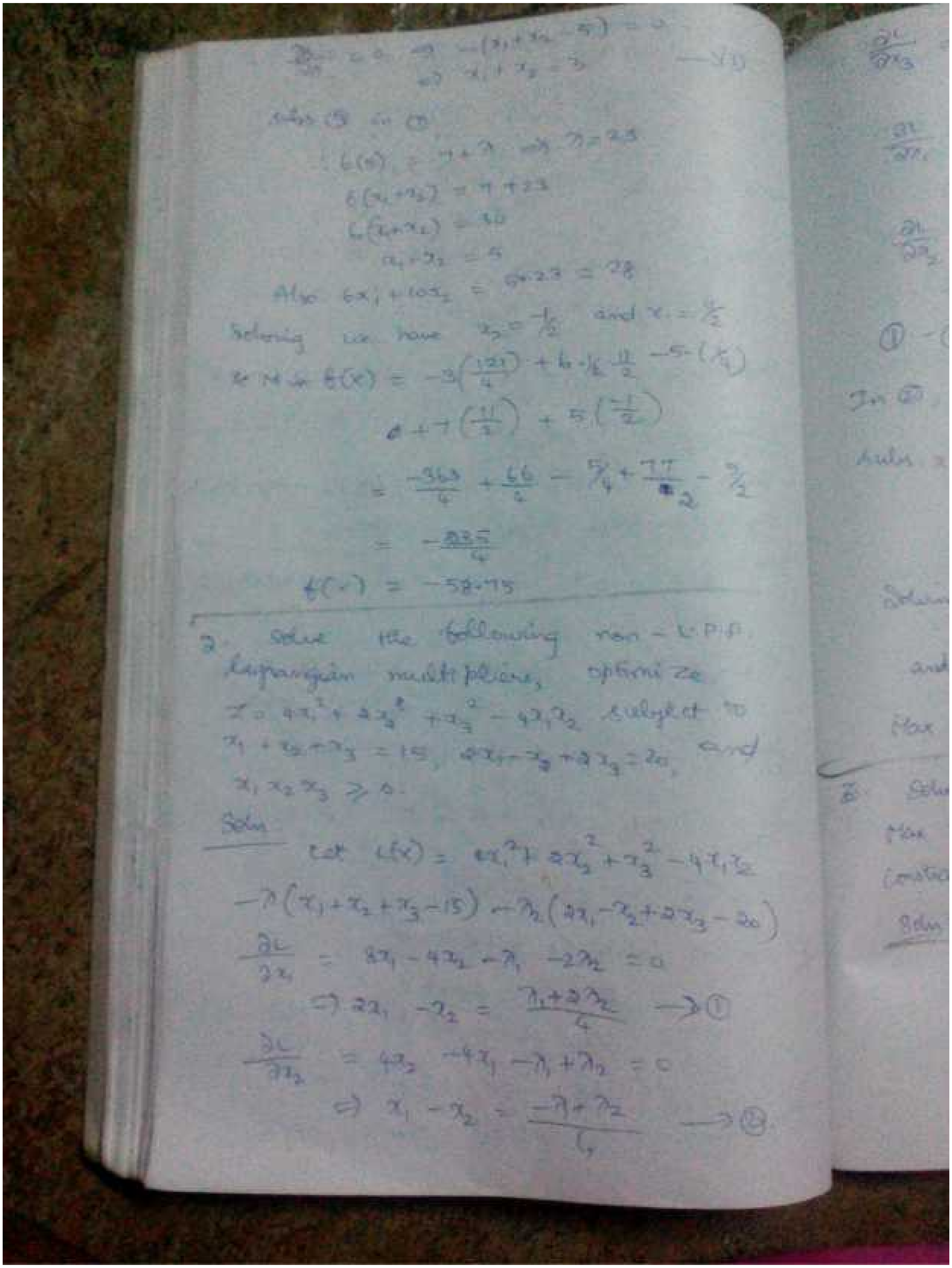
$\frac{\partial L}{\partial x_1} = 0 \Rightarrow -6x_1 - 6x_2 + 7 + \lambda = 0$

$6x_1 + 6x_2 = 7 + \lambda$

$6(x_1 + x_2) = 7 + \lambda \rightarrow (1)$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -6x_1 - 10x_2 + 5 + \lambda = 0$

$\Rightarrow 6x_1 + 10x_2 = 5 + \lambda \rightarrow (2)$



$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow -(x_1+x_2-5) = 0$

$\Rightarrow x_1 + x_2 = 5 \longrightarrow ③$

Subs ③ in ①,

$\therefore 6(5) = 7 + \lambda \Rightarrow \lambda = 23$

$6(x_1+x_2) = 7+23$

$6(x_1+x_2) = 30$

$x_1+x_2 = 5$

Also $6x_1 + 10x_2 = 5+23 = 28$

Solving we have $x_2 = -\frac{1}{2}$ and $x_1 = \frac{11}{2}$

$\therefore$ Max $f(x) = -3\left(\frac{121}{4}\right) + 6\cdot -\frac{1}{2}\cdot\frac{11}{2} - 5\cdot\left(\frac{1}{4}\right)$

$+ 7\left(\frac{11}{2}\right) + 5\left(\frac{-1}{2}\right)$

$= \frac{-363}{4} + \frac{66}{2} - \frac{5}{4} + \frac{77}{2} - \frac{5}{2}$

$= -\frac{235}{4}$

$f(x) = -58.75$

---

2. Solve the following non-L.P.P. Lagrangian multipliers, optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$ subject to $x_1 + x_2 + x_3 = 15$, $2x_1 - x_2 + 2x_3 = 20$, and $x_1, x_2\, x_3 \geq 0$.

Soln

Let $L(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

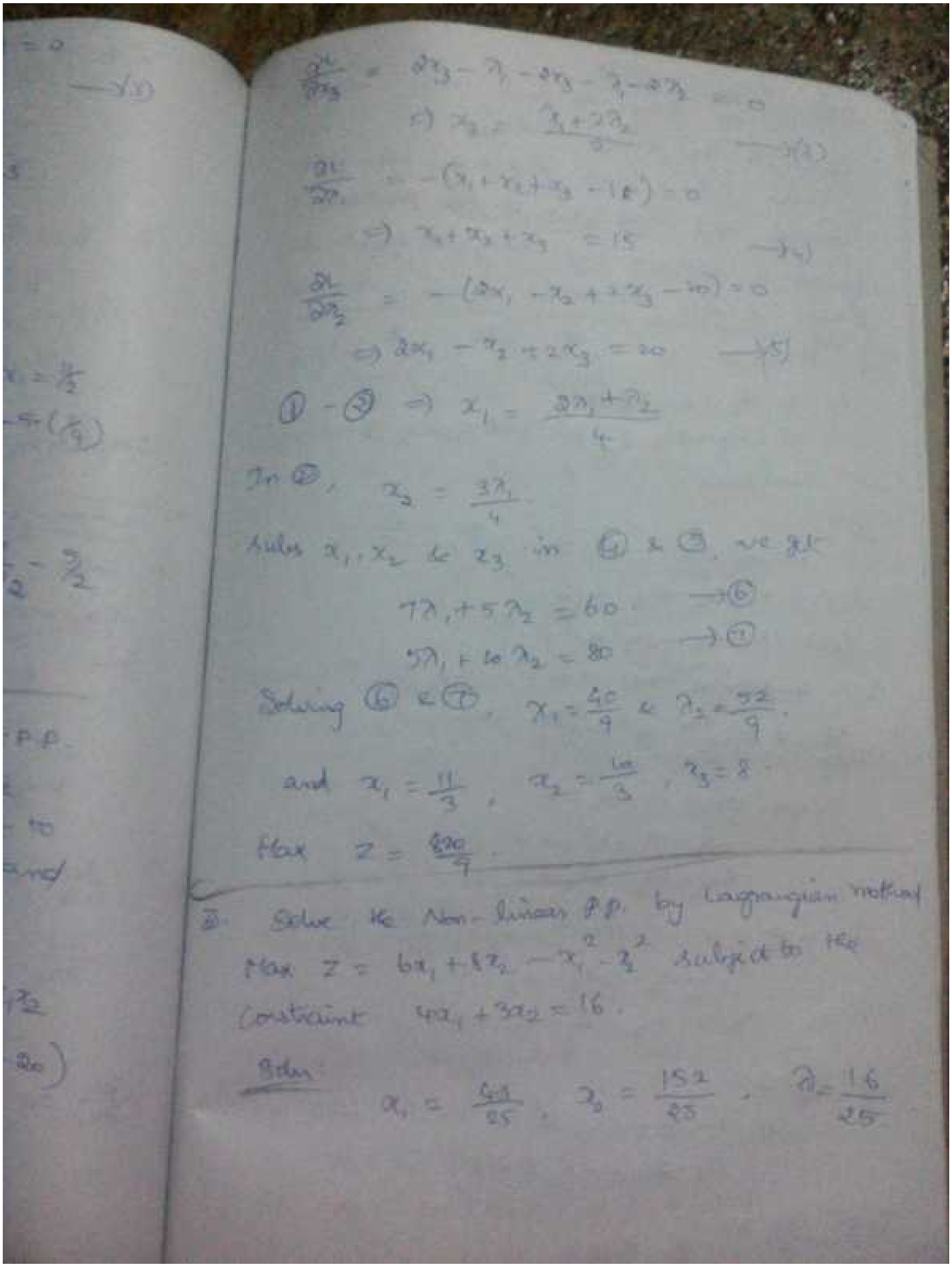
$-\lambda_1(x_1+x_2+x_3-15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$

$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$

$\Rightarrow 2x_1 - x_2 = \frac{\lambda_1 + 2\lambda_2}{4} \longrightarrow ①$

$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$

$\Rightarrow x_1 - x_2 = \frac{-\lambda_1 + \lambda_2}{4} \longrightarrow ②$



$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2x_3 - \lambda_1 - 2\lambda_2 = 0$

$\Rightarrow x_3 = \frac{\lambda_1 + 2\lambda_2}{2} \longrightarrow (3)$

$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0$

$\Rightarrow x_1 + x_2 + x_3 = 15 \longrightarrow (4)$

$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0$

$\Rightarrow 2x_1 - x_2 + 2x_3 = 20 \longrightarrow (5)$

① - ② $\Rightarrow x_1 = \frac{2\lambda_1 + \lambda_2}{4}$

In ②, $x_2 = \frac{3\lambda_1}{4}$

Subs x_1, x_2 & x_3 in ④ & ⑤, we get

$7\lambda_1 + 5\lambda_2 = 60 \longrightarrow$ ⑥

$5\lambda_1 + 10\lambda_2 = 80 \longrightarrow$ ⑦

Solving ⑥ & ⑦, $\lambda_1 = \frac{40}{9}$ & $\lambda_2 = \frac{52}{9}$

and $x_1 = \frac{11}{3}$, $x_2 = \frac{10}{3}$, $x_3 = 8$

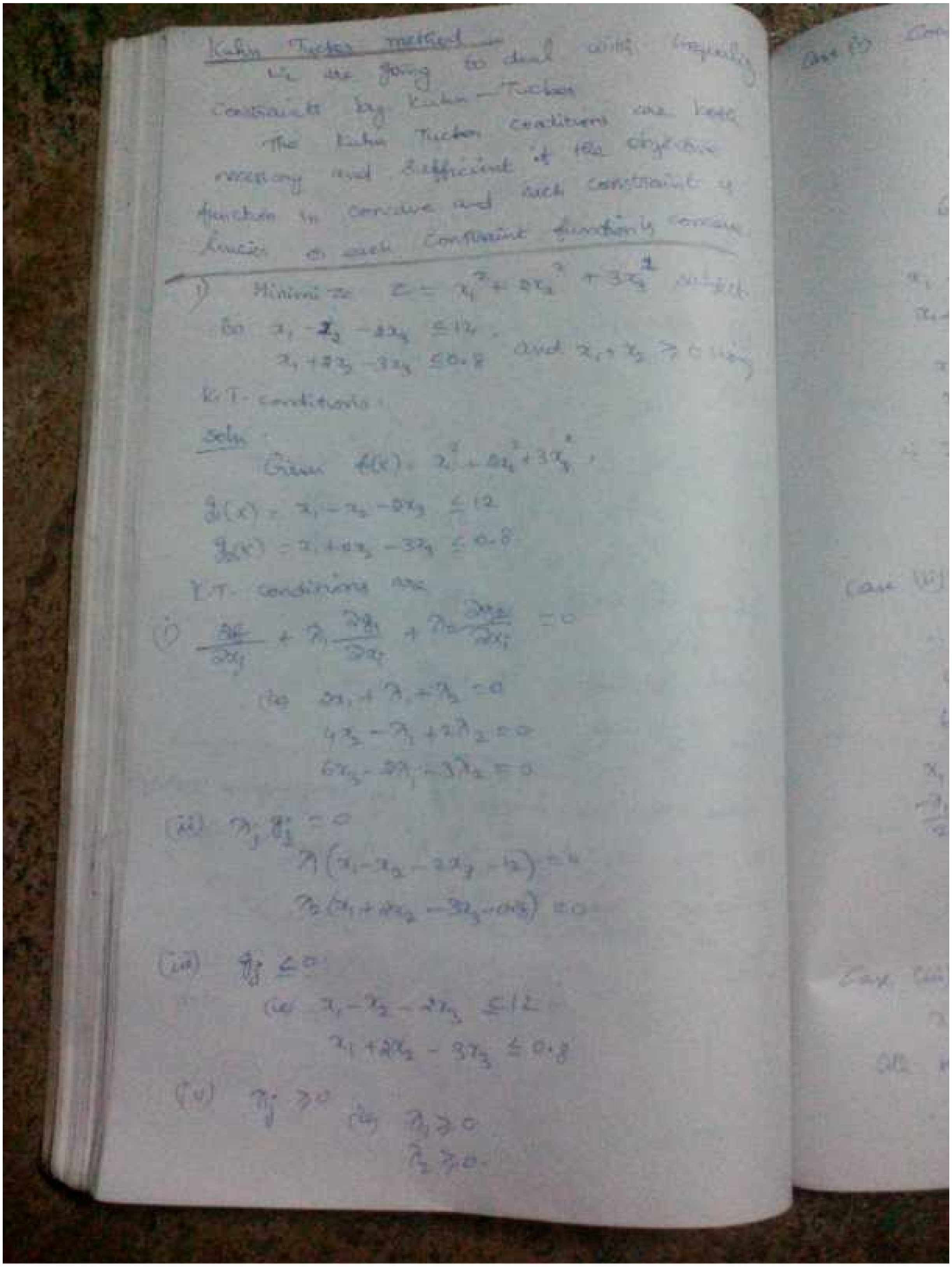
Max $Z = \frac{820}{9}$

II. Solve the Non-linear P.P. by Lagrangian method

Max $Z = 6x_1 + 8x_2 - x_1^2 - x_2^2$ subject to the constraint $4x_1 + 3x_2 = 16$.

Soln:

$x_1 = \frac{48}{25}$, $x_2 = \frac{152}{25}$, $\lambda = \frac{16}{25}$



Kuhn Tucker method

We are going to deal with inequality constraints by Kuhn-Tucker conditions are here.

The Kuhn Tucker conditions necessary and sufficient if the objective function is concave and each constraint is linear or each constraint function is convex.

1) Minimize $Z = x_1^2 + 2x_2^2 + 3x_3^2$ subject to $x_1 - x_2 - 2x_3 \le 12$,
$x_1 + 2x_2 - 3x_3 \le 0.8$ and $x_1, x_2 \ge 0$ using K.T. conditions.

Soln:

Given $f(x) = x_1^2 + 2x_2^2 + 3x_3^2$,

$g_1(x) = x_1 - x_2 - 2x_3 \le 12$

$g_2(x) = x_1 + 2x_2 - 3x_3 \le 0.8$

K.T. conditions are

(i) $\frac{\partial f}{\partial x_j} + \lambda_1 \frac{\partial g_1}{\partial x_j} + \lambda_2 \frac{\partial g_2}{\partial x_j} = 0$

(ie) $2x_1 + \lambda_1 + \lambda_2 = 0$

$4x_2 - \lambda_1 + 2\lambda_2 = 0$

$6x_3 - 2\lambda_1 - 3\lambda_2 = 0$

(ii) $\lambda_j g_j = 0$

$\lambda_1(x_1 - x_2 - 2x_3 - 12) = 0$

$\lambda_2(x_1 + 2x_2 - 3x_3 - 0.8) = 0$

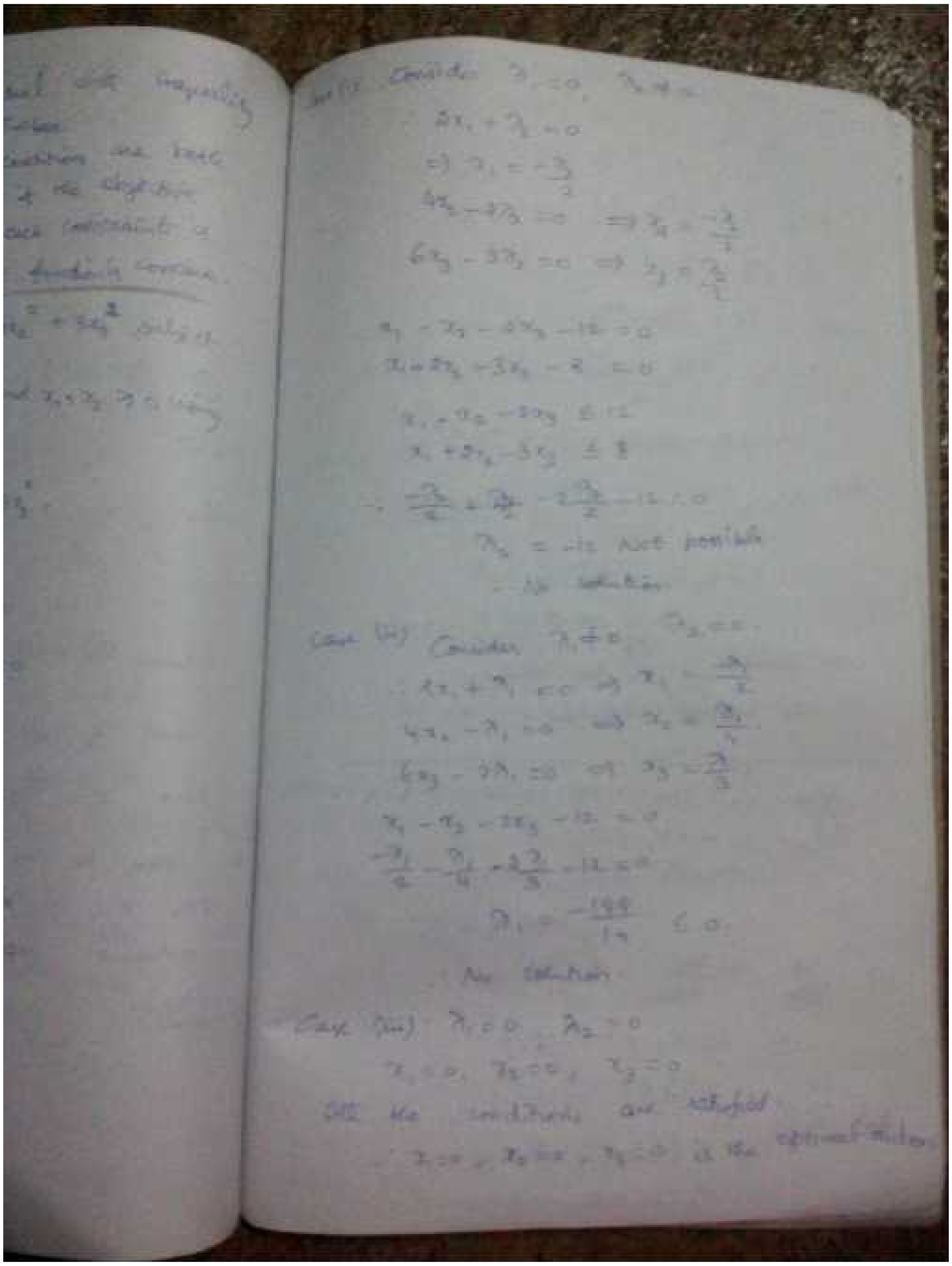
(iii) $g_j \le 0$

(ie) $x_1 - x_2 - 2x_3 \le 12$

$x_1 + 2x_2 - 3x_3 \le 0.8$

(iv) $\lambda_j \ge 0$ (ie) $\lambda_1 \ge 0$

$\lambda_2 \ge 0$.



Case (i) Consider $\lambda_1 = 0$, $\lambda_2 \neq 0$

$\therefore 2x_1 + \lambda_2 = 0$

$\Rightarrow x_1 = -\frac{\lambda_2}{2}$

$4x_2 - \lambda_2 = 0 \Rightarrow x_2 = -\frac{\lambda_2}{4}$

$6x_3 - 3\lambda_2 = 0 \Rightarrow x_3 = \frac{\lambda_2}{2}$

$x_1 - x_2 - 2x_3 - 12 = 0$

$x_1 + 2x_2 - 3x_3 - 8 = 0$

$x_1 - x_2 - 2x_3 \leq 12$

$x_1 + 2x_2 - 3x_3 \leq 8$

$\therefore -\frac{\lambda_2}{2} + \frac{\lambda_2}{4} - 2\frac{\lambda_2}{2} - 12 = 0$

$\lambda_2 =$ [illegible] Not possible

$\therefore$ No solution

Case (ii) Consider $\lambda_1 \neq 0$, $\lambda_2 = 0$

$\therefore 2x_1 + \lambda_1 = 0 \Rightarrow x_1 = -\frac{\lambda_1}{2}$

$4x_2 - \lambda_1 = 0 \Rightarrow x_2 = \frac{\lambda_1}{4}$

$6x_3 - 2\lambda_1 = 0 \Rightarrow x_3 = \frac{\lambda_1}{3}$

$x_1 - x_2 - 2x_3 - 12 = 0$

$-\frac{\lambda_1}{2} - \frac{\lambda_1}{4} - \frac{2\lambda_1}{3} - 12 = 0$

$\lambda_1 = -\frac{144}{17} \leq 0$

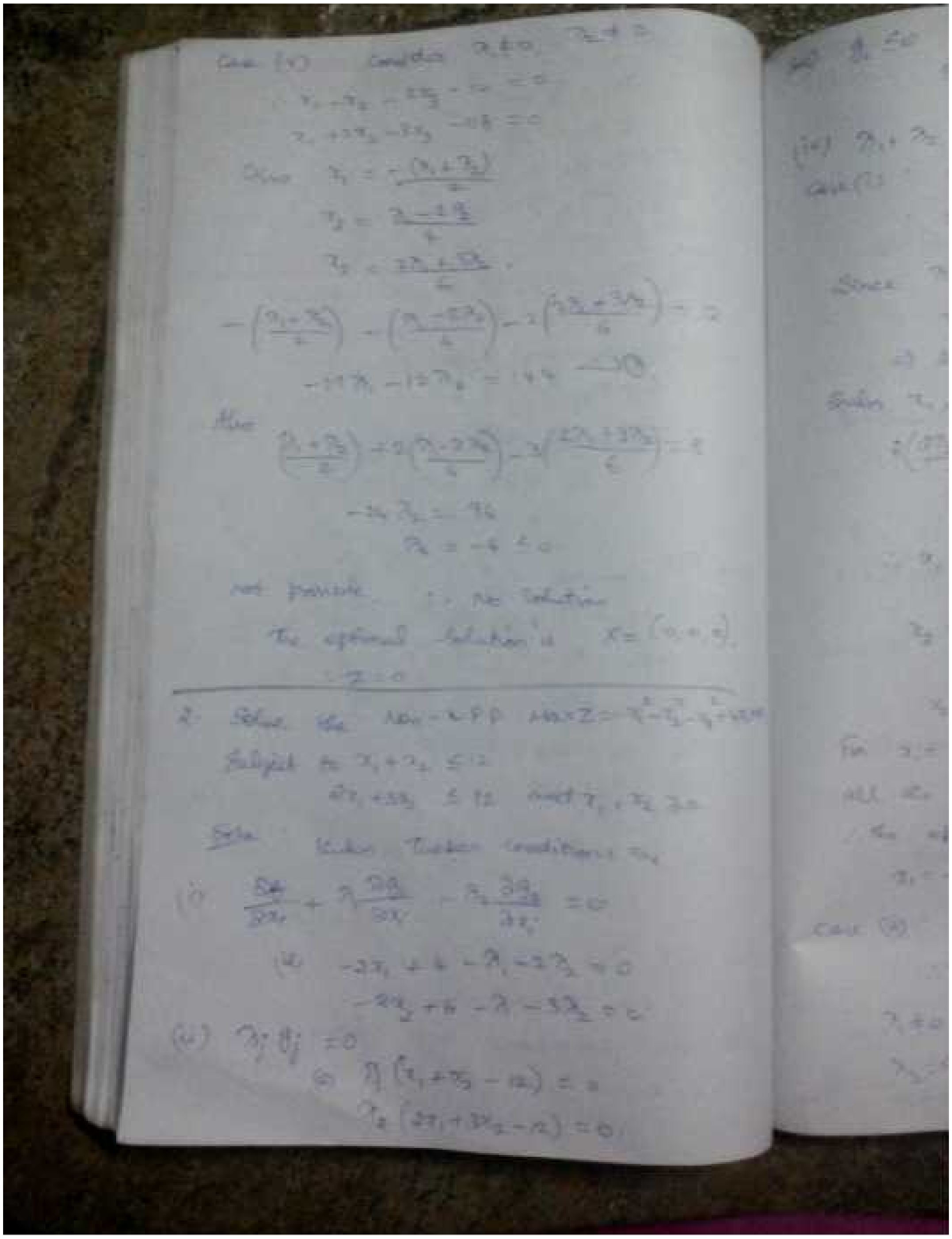
$\therefore$ No solution

Case (iii) $\lambda_1 = 0$, $\lambda_2 = 0$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$

All the conditions are satisfied

$\therefore x_1 = 0$, $x_2 = 0$, $x_3 = 0$ is the optimal solution.



Case (x) Condition $\lambda_1 \neq 0, \lambda_2 \neq 0$

$x_1 - x_2 - 2x_3 - [illegible] = 0$

$x_1 + 2x_2 - 3x_3 - 8 = 0$

Since $x_1 = \frac{-(\lambda_1+\lambda_2)}{2}$

$x_2 = \frac{\lambda_1 - 2\lambda_2}{4}$

$x_3 = \frac{2\lambda_1 + 3\lambda_2}{6}$

$-\left(\frac{\lambda_1+\lambda_2}{2}\right) - \left(\frac{\lambda_1-2\lambda_2}{4}\right) - 2\left(\frac{2\lambda_1+3\lambda_2}{6}\right) = 2$

$-17\lambda_1 - 12\lambda_2 = 24 \longrightarrow (1)$

Also

$\left(\frac{\lambda_1+\lambda_2}{2}\right) + 2\left(\frac{\lambda_1-2\lambda_2}{4}\right) - 3\left(\frac{2\lambda_1+3\lambda_2}{6}\right) = 8$

$-24\lambda_2 = 96$

$\lambda_2 = -4 \le 0$

not possible ∴ No Solution

The optimal solution is $X = (0, 0, 0)$

$Z = 0$

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2. Solve the Non-L.P.P Max $Z = -x_1^2 - x_2^2 - [illegible]$

Subject to $x_1 + x_2 \le 12$

$2x_1 + 3x_2 \le 12$ and $x_1, x_2 \ge 0$

Soln: Kuhn Tucker conditions are

(i) $\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} - \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$

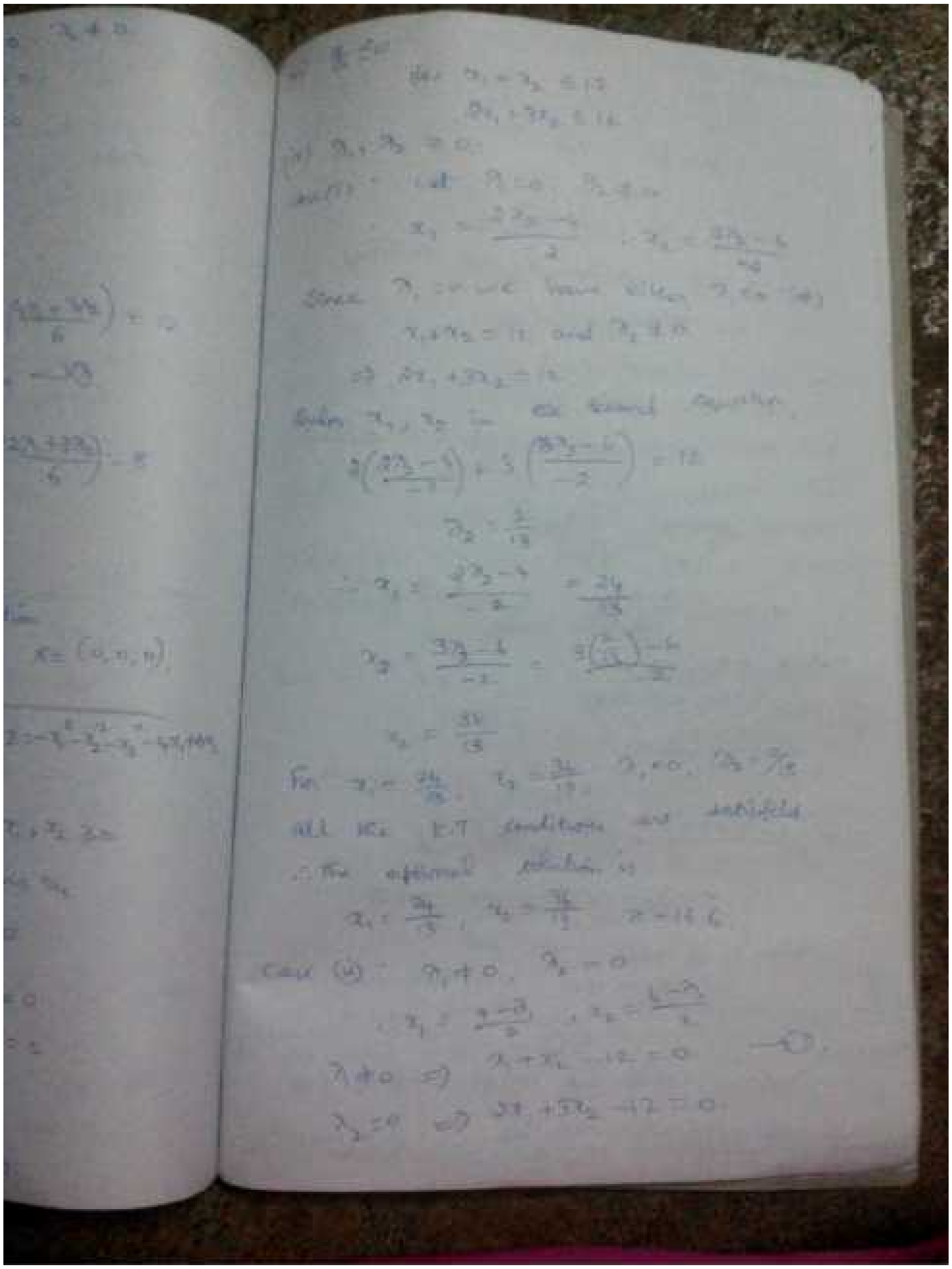
(ie) $-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$

$-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$

(ii) $\lambda_j g_j = 0$

(ie) $\lambda_1(x_1 + x_2 - 12) = 0$

$\lambda_2(2x_1 + 3x_2 - 12) = 0$



Since $\lambda_1 = 0$ we have ...

$x_1 + x_2 = 12$ and $\lambda_2 \neq 0$

$\Rightarrow 2x_1 + 3x_2 = 12$

$x_1 = \frac{24}{13}$

$x_2 = \frac{36}{13}$

All the K.T conditions are satisfied

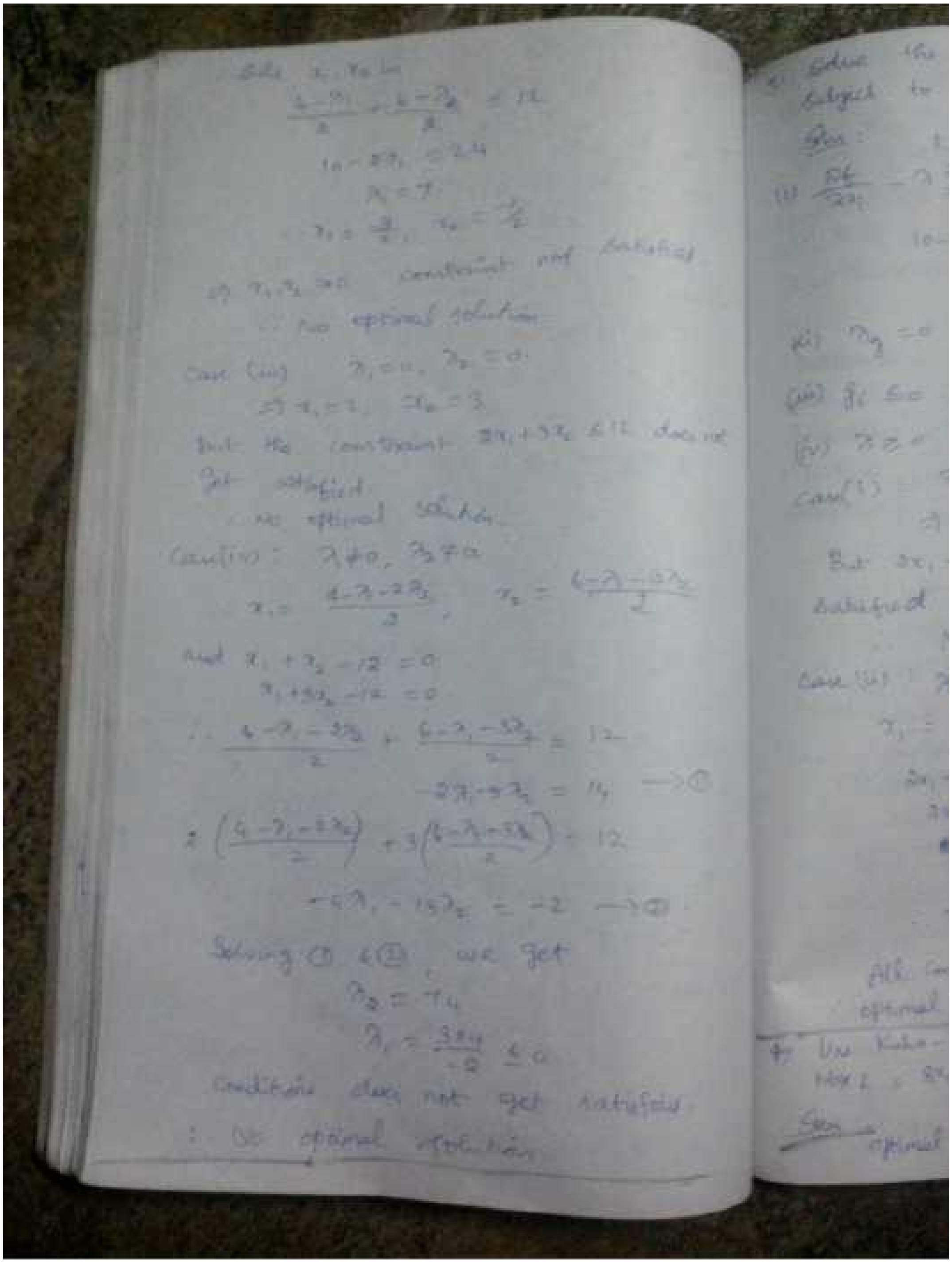
$\therefore$ The optimal solution is

$x_1 = \frac{24}{13}, \quad x_2 = \frac{36}{13}$

Case (iv): $\lambda_1 \neq 0, \quad \lambda_2 = 0$

$\lambda_1 \neq 0 \Rightarrow x_1 + x_2 - 12 = 0$ —①

$



Sub x_1, x_2 in

$$\frac{6-\lambda_1}{2} + \frac{6-\lambda_2}{2} = 12$$

$$12 - 2\lambda_1 = 24$$

$$\lambda_1 = 7$$

$$\therefore x_1 = \frac{3}{2}, \quad x_2 = \frac{3}{2}$$

$\Rightarrow x_1, x_2 > 0$ constraint not satisfied

$\therefore$ no optimal solution.

Case (iii) $\lambda_1 = 0, \lambda_2 = 0$

$\Rightarrow x_1 = 2, \quad x_2 = 3$

But the constraint $2x_1 + 3x_2 \le 12$ does not get satisfied.

$\therefore$ no optimal solution.

Case (iv): $\lambda_1 \neq 0, \lambda_2 \neq 0$

$$x_1 = \frac{4-\lambda_1-2\lambda_2}{2}, \quad x_2 = \frac{6-\lambda_1-3\lambda_2}{2}$$

And $x_1 + x_2 - 12 = 0$

$x_1 + 3x_2 - 12 = 0$

$$\therefore \frac{4-\lambda_1-2\lambda_2}{2} + \frac{6-\lambda_1-3\lambda_2}{2} = 12$$

$$-2\lambda_1 - 5\lambda_2 = 14 \rightarrow ①$$

$$2\left(\frac{4-\lambda_1-2\lambda_2}{2}\right) + 3\left(\frac{6-\lambda_1-3\lambda_2}{2}\right) = 12$$

$$-5\lambda_1 - 15\lambda_2 = -2 \rightarrow ②$$

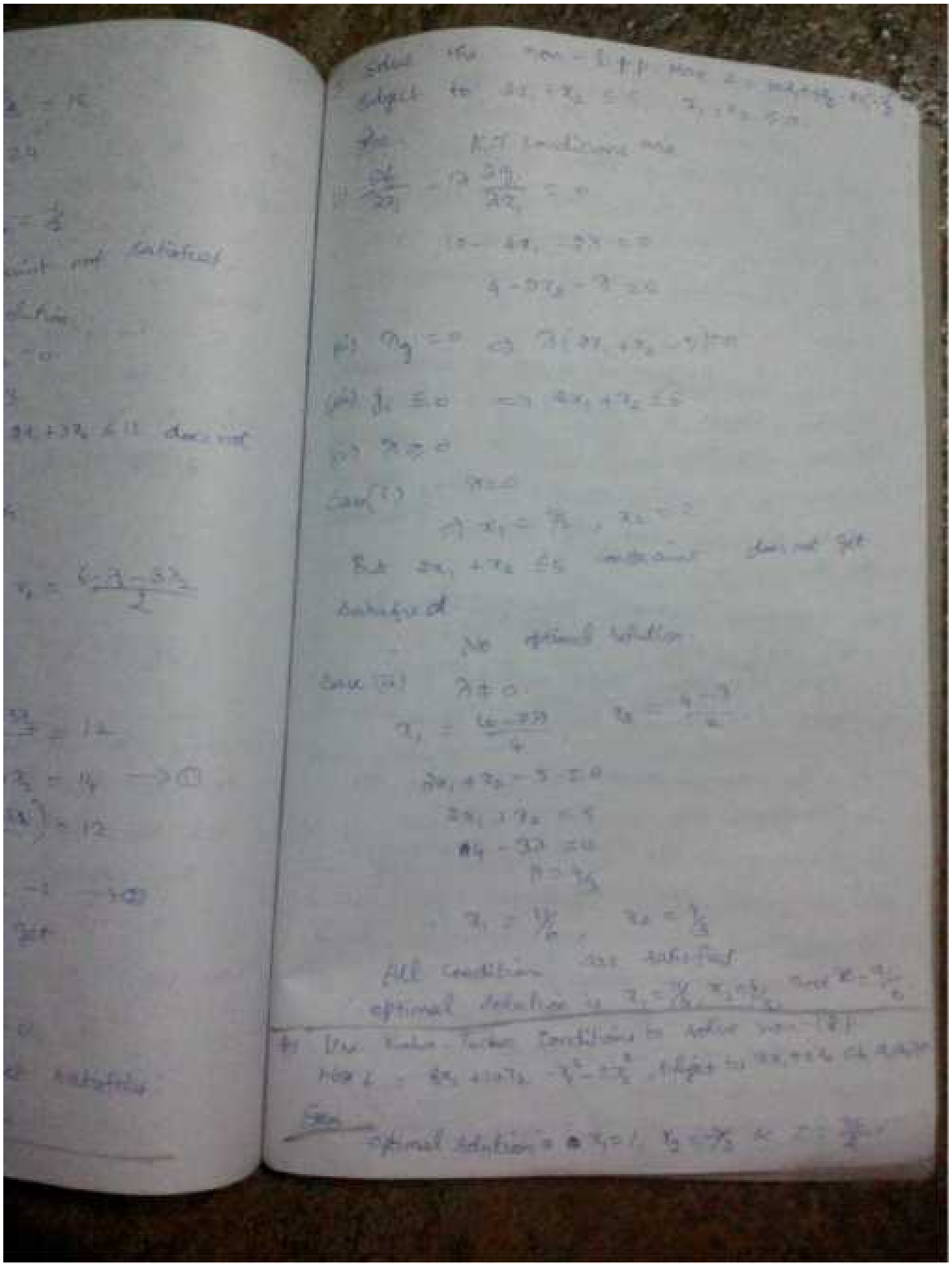
Solving ① & ②, we get

$$\lambda_2 = 74$$

$$\lambda_1 = \frac{384}{-2} \le 0$$

Condition does not get satisfied.

$\therefore$ no optimal solution.



Solve the non-L.P.P Max $Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \le 5$, $x_1, x_2 \ge 0$

Soln: K.T conditions are

(i) $\frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$

$10 - 4x_1 - 2\lambda = 0$

$4 - 2x_2 - \lambda = 0$

(ii) $\lambda g = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0$

(iii) $g \le 0 \Rightarrow 2x_1 + x_2 \le 5$

(iv) $\lambda \ge 0$

Case (i) $\lambda = 0$

$\Rightarrow x_1 = 5/2$, $x_2 = 2$

But $2x_1 + x_2 \le 5$ constraint does not get satisfied

No optimal solution.

Case (ii) $\lambda \ne 0$

$x_1 = \frac{10 - 2\lambda}{4}$, $x_2 = \frac{4 - \lambda}{2}$

$2x_1 + x_2 - 5 = 0$

$2x_1 + x_2 = 5$

$4 - 3\lambda = 0$

$\lambda = 4/3$

$x_1 = 11/6$, $x_2 = 4/3$

All condition are satisfied

optimal solution is $x_1 = 11/6$, $x_2 = 4/3$ and $Z = 91/6$

Use Kuhn-Tucker conditions to solve non-LPP Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$, subject to $3x_1 + 2x_2 \le 6$, $x_1, x_2 \ge 0$

Soln

optimal solution = $x_1 = 1$, $x_2 = 3/2$ & $Z =$ [illegible]